A CLASS OF SIGNED-RANK TESTS FOR THE ONE-SAMPLE LOCATION PROBLEM WITH APPLICATION TO SIMPLE REPEATED-MEASURES DESIGNS

By

ZIYAD R. MAHFOUD

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

2001

I dedicate this work to my parents, Riyad and Guilda and to my brother George.

ACKNOWLEDGMENTS

I am very thankful to Dr. Ronald H. Randles for being my dissertation advisor and for directing me with my studies throughout the last three years. His guidance, encouragement, wisdom and kindness have made this work possible. I would like also to thank the members of my committee, Dr. Malay Ghosh, Dr. James L. Kepner, Dr. James Keesling and Dr. Pejaver V. Rao, for their helpful remarks.

I am very grateful to my parents for all their support and encouragement, for their love and understanding, and for believing in me. I wish to give a heartful thanks to my brother for his unconditional love and patience. I thank my cousin Tony for being near when I needed advice, my relatives in Jacksonville for being my family when I was away from home.

I also would like to acknowledge my friends in Lebanon for all the good times that they showed me when I went visiting, my friends in the United States of America for making my stay at Gainesville a cherishable one.

Finally, I wish to give special thanks to my friend, Mahfoud, for being thoughtful, caring, and supportive.

TABLE OF CONTENTS

page

ACKNOWLEDGMENTS iii

LIST OF TABLES vi

ABSTRACT vii

CHAPTERS

1 INTRODUCTION 1

2 A MULTIVARIATE ONE-SAMPLE TEST FOR LOCATION . . . 11

- 2.1 Definition of the Test Statistic 11
- 2.2 The Choice of the ϕ Function 13
- 2.3 Affine-Invariance of W_{np} 14
- 2.4 The Asymptotic Null Distribution of W_{np} 16

3 TEST COMPARISONS 24

- 3.1 Asymptotic Distribution of W_{np} Under Contiguous Alternatives 24
- 3.2 ARE of W_{np} 30
- 3.3 Monte Carlo Results 36

4 SIMPLE REPEATED-MEASURES DESIGNS 44

5 MULTIVARIATE SIGN AND SIGNED-RANK TESTS APPLIED TO THE SIMPLE REPEATED-MEASURES DESIGNS 53

- 5.1 Introduction 53
- 5.2 The Test Statistics 54
- 5.3 Asymptotic Null Distribution 56

6 TEST COMPARISONS 59

- 6.1 Asymptotic Distribution Under Contiguous Alternatives . . . 59
- 6.2 Asymptotic Relative Efficiencies 64
- 6.3 Monte Carlo Results 71

7 SUMMARY, CONCLUSIONS, AND FUTURE RESEARCH 79

- 7.1 Summary and Conclusions 79

7.2 Future Research 80

APPENDICES 81

A COMPUTING $\hat{\mathbf{A}}$ 81

B UNDERLYING DISTRIBUTIONS AND RELATIONS AMONG THEM 82

REFERENCES 85

BIOGRAPHICAL SKETCH 90

LIST OF TABLES

Table page

3.1 Asymptotic Relative Efficiencies Under the Power Family of Distributions . . . 34

3.2 Asymptotic Relative Efficiencies Under Multivariate *t*-Distribution . . 35

3.3 Monte Carlo Results for the Bivariate Power Family of Distributions . 40

3.4 Monte Carlo Results for the Bivariate t-Distributions . . . 41

3.5 Monte Carlo Results for the Multivariate Power Family of Distributions 42

3.6 Monte Carlo Results for the Multivariate t-Distributions . . . 43

6.1 Asymptotic Relative Efficiencies Under the Power Family of Distributions for the Repeated-Measures Problem . . . 68

6.2 Asymptotic Relative Efficiencies Under the Multivariate t-Distributions for the Repeated-Measures Problem . . . 69

6.3 Monte Carlo Results for the Power Family, with $\mathbf{\Sigma}=\mathbf{I}_4$ . . . 75

6.4 Monte Carlo Results for the Power Family, with $\mathbf{\Sigma}=\mathbf{E}\mathbf{E}'$ . . . 76

6.5 Monte Carlo Results for the Family of *t*-Distributions, with $\mathbf{\Sigma}=\mathbf{I}_4$ . . . 77

6.6 Monte Carlo Results for the Family of *t*-Distributions, with $\mathbf{\Sigma}=\mathbf{E}\mathbf{E}'$. 78

Abstract of Dissertation Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Doctor of Philosophy

A CLASS OF SIGNED-RANK TESTS FOR THE ONE-SAMPLE LOCATION PROBLEM WITH APPLICATION TO SIMPLE REPEATED-MEASURES DESIGNS

By

Ziyad R. Mahfoud

August, 2001

Chair: Ronald H. Randles
Major Department: Statistics

A class of affine-invariant signed-rank test statistics is proposed for the one-sample multivariate location problem. This class is based on the transformation-retransformation approach and a directional transformation. It contains several test statistics that have been previously proposed for this problem. For the class of symmetric distributions, the asymptotic null distribution of the proposed class of statistics is shown to be chi-square with p degrees of freedom, where p represents the dimension of the observed vectors. In addition, the asymptotic distribution of the test statistic, under a sequence of alternatives approaching the null, is obtained. The Pitman asymptotic relative efficiency of this class with respect to Hotelling's T^2 is established. One particular member of this class of tests is recommended. This specific statistic is obtained by using a linear function of the ranks of Euclidean norms of the transformed vectors. Comparisons of this new test statistic and several leading competitors are made through efficiency calculations and Monte Carlo studies. The new statistic has strong efficiencies over a wide spectrum of distributions, ranging from very light-tailed distributions to very heavy-tailed ones. It performs as well as and in many cases better than its competitors. Yet, the best feature of this statistic is that it is very easy to compute for data in any practical dimension. This distinguishes it from the other affine-invariant signed-rank tests in the literature.

This new signed-rank test is also applied to a simple repeated-measures setting. Since the test is affine-invariant, it is not affected by the structure of the variance-covariance matrix of the underlying population, as is the case with some procedures in the literature. Once again, the particular signed-rank statistic developed in this dissertation showed strong efficiencies and performed very well in comparison with other well-known procedures over a wide variety of distributional assumptions.

CHAPTER 1
INTRODUCTION

This dissertation investigates test statistics for the one-sample multivariate location problem. A new multivariate signed-rank procedure is proposed and its application to certain repeated-measures designs is studied. The one-sample problem is introduced in this chapter while the repeated-measures model is presented in Chapter 4.

For the one-sample location problem, assume that $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are independently and identically distributed (iid) as $\mathbf{Y}$, where $\mathbf{Y} = (Y_1, \cdots, Y_p)'$ has an absolutely continuous p-dimensional distribution with location parameter $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_p)'$. We wish to test

$$H_o : \boldsymbol{\theta} = \mathbf{0} \quad \text{against} \quad H_a : \boldsymbol{\theta} \neq \mathbf{0}. \tag{1.1}$$

The value $\mathbf{0}$ is used without loss of generality, since if the null was $\boldsymbol{\theta} = \boldsymbol{\theta}_0$, the test of (1.1) would be performed on the random variables $\mathbf{Y}_1 - \boldsymbol{\theta}_0, \cdots, \mathbf{Y}_n - \boldsymbol{\theta}_0$.

Probably the best known parametric procedure for this setting is Hotelling's (1931) T^2, which is based on the statistic

$$T^2 = T^2(\mathbf{Y}_1, \cdots, \mathbf{Y}_n) = n\bar{\mathbf{Y}}'(\mathbf{S})^{-1}\bar{\mathbf{Y}},$$

where

$$\bar{\mathbf{Y}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{Y}_i \quad \text{and} \quad \mathbf{S} = \frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{Y}_i - \bar{\mathbf{Y}})(\mathbf{Y}_i - \bar{\mathbf{Y}})'.$$

Under H_o, if $\boldsymbol{\Sigma}$, the variance-covariance matrix of $\mathbf{Y}$, is positive definite, then T^2 has an asymptotic chi-square distribution with p-degrees of freedom (χ^2_p). When

$\mathbf{Y}$ has a distribution that is p-variate normal, T^2 is the likelihood ratio test of (1.1) and the null distribution of T^2 becomes a multiple of an F-distribution. In particular, H_0 is rejected in favor of H_a, if $T^2 \geq \frac{(n-1)p}{n-p} F_{p,n-p}(\alpha)$, where $F_{\eta_1,\eta_2}(\alpha)$ is the upper αth quantile of the F-distribution with η_1 numerator degrees of freedom and η_2 denominator degrees of freedom. This procedure is invariant with respect to nonsingular linear transformations of the observations. That is, the test statistic satisfies:

$$T^2(\mathbf{y}_1, \cdots, \mathbf{y}_n) = T^2(\mathbf{D}\mathbf{y}_1, \cdots, \mathbf{D}\mathbf{y}_n) \tag{1.2}$$

for every $(\mathbf{y}_1, \cdots, \mathbf{y}_n)$ and all nonsingular $p \times p$ matrices $\mathbf{D}$. This property ensures that the value of the test statistic remains unchanged following rotations of the observations about the origin, reflections of the observations about a $(p-1)$-dimensional hyper-plane through the origin, or changes in scale. This desirable invariance property will be called affine-invariance.

Early competitors of Hotelling's T^2 were restricted to the bivariate case $(p = 2)$. Hodges (1955) and Blumen (1958) proposed bivariate affine-invariant sign tests that are distribution-free for the class of distributions that are directionally symmetric (i.e., under H_o, $\frac{\mathbf{Y}_i}{||\mathbf{Y}_i||} \overset{d}{=} -\frac{\mathbf{Y}_i}{||\mathbf{Y}_i||}$, where $|| \cdot ||$ represents the Euclidean norm). Blumen showed that under the null hypothesis his test has an asymptotic chi-square distribution with 2 degrees of freedom. Joffe and Klotz (1962) gave expressions for the exact and asymptotic null distributions of Hodges' test. They also calculated the Bahadur approximate efficiency of Hodges' test with respect to Hotelling's T^2 when the underlying distribution of the $\mathbf{Y}$'s is the bivariate normal distribution. For $8 \leq n \leq 12$, Koltz (1964) computed exact powers of the Blumen and Hodges' tests under normal location alternatives and efficiencies of Hodges' test relative to Hotelling's T^2. Bennett (1962 and 1964) proposed two new tests. The first procedure is based on the frequencies associated with the signs of the components. The second procedure is a quadratic form based on the component-wise Wilcoxon's signed-rank statistic. Both tests have a χ_2^2 asymptotic null distribution. Bennett's tests are not affine-invariant. Chatterjee (1966) also proposed a bivariate distribution-free test that has a χ_2^2 limiting null distribution. Killeen and Hettmansperger (1972) proposed a new bivariate signed rank test that is also a generalization of Wilcoxon's signed-rank test. They made an exact Bahadur efficiency comparison of Hotelling's T^2 with respect to the new test, Blumen's test and Hodges's test. Brown and Hettmansperger (1987) proposed an affine-invariant bivariate signed-rank test based on the median of Oja (1983). Brown and Hettmansperger (1989) also developed an affine-invariant bivariate version of the sign test.

For the more general case, $p \geq 1$, many nonparametric analogs of Hotelling's T^2 have been proposed. Bickel (1965) proposed using quadratic forms involving coordinate-wise sign and Wilcoxon signed-rank statistic respectively. Sen and Puri (1967) suggested a more general approach by forming

$$B_j = \sum_{i=1}^{n} E_{ij} C_{ij}$$

where E_{ij} is a general score function applied to the rank of $|Y_{ij}|$ among $|Y_{1j}|, \cdots, |Y_{nj}|$, and $C_{ij} = +1$ or -1 according to whether $Y_{ij} > 0$ or < 0 respectively. Then the test is based on

$$S_n = \frac{1}{n}\mathbf{B}(\mathbf{W}^{-1})\mathbf{B}',$$

where $\mathbf{B} = (B_1, \cdots, B_p)$ and $\mathbf{W} = (w_{jj'})$, $w_{jj'} = \sum_{i=1}^{n} E_{ij} E_{ij'} C_{ij} C_{ij'}$. The limiting distribution of S_n, under the null hypothesis, is χ_p^2. Unlike Hotelling's T^2, these tests are not affine-invariant. Thus, their power and efficiency depend on the direction of the shift of $\boldsymbol{\theta}$ from the origin and the variance-covariance matrix. Dietz (1982) tried to create an affine-equivariant test procedure for the bivariate case by means of a three-step procedure. First, a certain transformation is applied to the observations. Second, new coordinate axes are chosen. Third, Bickel's coordinate-wise sign and signed-rank tests are applied on the transformed and rotated observations. This multi-step process was created to achieve affine-invariance with a test procedure based on coordinate-wise statistics. Dietz (1984) extended those results to general linear rank tests for the multivariate location problem. In this extension, step two is dropped and the resulting statistics are no longer invariant under nonsingular linear transformations. However, for elliptically symmetric contiguous alternatives, the asymptotic efficiencies of the Dietz test are independent of the direction of shift and the underlying variance-covariance matrix; moreover, its efficiencies are equal to those of the corresponding univariate linear signed-rank tests.

Many affine-invariant nonparametric multivariate procedures have been proposed for this problem. We will mention most of the tests with a detailed discussion for tests which are extensions of the univariate Wilcoxon signed rank test.

Randles (1989) introduced a multivariate affine-invariant sign test based on counts called interdirections. Randles' test includes both the univariate sign test and Blumen's bivariate sign test as special cases. It has small-sample distribution free property and if the observations are from distributions with elliptical directions (see appendix B for definitions of various classes of distributions and relationships between them), then the null limiting distribution of the test is χ^2_p. This test performs well compared with Hotelling's T^2 for heavy-tailed distributions but not as well for light-tailed distributions. Because of the complicated nature of these counts, the test's calculation becomes more tedious when the dimension and/or the sample size increase.

Hettmansperger, Nyblom and Oja (1994) extended the Brown and Hettmansperger bivariate sign test to the general multivariate case. For elliptical distributions, the Pitman efficiencies of their test statistic are equivalent to those of Randles' (1989) sign test. Hettmansperger, Möttönen and Oja (1997) suggested an affine-invariant signed-rank test based on the generalized median of Oja. Their work extends some of the bivariate rank methods discussed by Brown and Hettmansperger (1987). They assumed that the $\mathbf{Y}$s come from a p-variate continuous distribution that is symmetric around $\boldsymbol{\theta}$. To be able to describe their test statistic, let

$$\mathbf{V}_{\mathbf{k}}(\boldsymbol{\theta}) = \mathbf{V}_{\mathbf{k}}(\mathbf{y}_{k_1}, \cdots, \mathbf{y}_{k_p}, \boldsymbol{\theta}) = \frac{1}{p!}\text{abs}\left\{\begin{vmatrix} 1 & \cdots & 1 & 1 \\ \mathbf{y}_{k_1} & \cdots & \mathbf{y}_{kp} & \boldsymbol{\theta} \end{vmatrix}\right\}$$

denote the volume of the simplex determined by $\mathbf{y}_{k_1}, \cdots, \mathbf{y}_{k_p}$ along with $\boldsymbol{\theta}$. Here abs{} indicates the absolute value. Let K be the set of all the $N_K = \binom{n}{p}$ different p-tuples chosen from the integers $1, \cdots, n$. Also, $\mathbf{k} = (k_1, \cdots, k_p) \in K$ represents a p-subset from the original observations. Note that Oja's median (1983) is the vector $\hat{\boldsymbol{\theta}}$ that minimizes $D_n(\boldsymbol{\theta}) = N_K^{-1} \sum_{\mathbf{k} \in K} \mathbf{V}_{\mathbf{k}}(\boldsymbol{\theta})$. Hettmansperger, Möttönen and Oja defined the "rank" of $\mathbf{y}_i$ as

$$\mathbf{R}_n(\mathbf{y}_i) = N_K^{-1} \sum_{\mathbf{k} \in K} \mathbf{Q}_{\mathbf{k}}(\mathbf{y}_i),$$

where

$$\mathbf{Q}_{\mathbf{k}}(\mathbf{y}_i) = p! \nabla \mathbf{V}_{\mathbf{k}}(\mathbf{y}_i)$$

and $\nabla \mathbf{V}_{\mathbf{k}}(\mathbf{y})$ is the gradient of $\mathbf{V}_{\mathbf{k}}(\boldsymbol{\theta})$ evaluated at $\boldsymbol{\theta} = \mathbf{y}$. They showed that $\mathbf{R}_n(\hat{\boldsymbol{\theta}}) = \mathbf{0}$ and that a sign test of $H_o : \boldsymbol{\theta} = \mathbf{0}$ can be based on a quadratic form in $\mathbf{R}_n(\mathbf{0})$. The signed-rank of $\mathbf{y}_i$ is obtained in the following manner. Let A be the set of all 2^p possible vectors $(\pm 1, \cdots, \pm 1)$ and define

$$\mathbf{Q}_{\mathbf{k}}^{+}(\mathbf{y}_i) = 2^{-p} \sum_{\mathbf{a} \in A} p! \nabla \mathbf{V}_{\mathbf{ak}}(\mathbf{y}_i).$$

where $\mathbf{a} = (a_1, \cdots, a_p)$ and $\mathbf{V}_{\mathbf{ak}}(\mathbf{y}) = \mathbf{V}_{\mathbf{k}}(a_1 \mathbf{y}_{k1}, \cdots, a_p \mathbf{y}_{k_p}, \mathbf{y})$. The signed-rank of $\mathbf{y}_i$ is then

$$\mathbf{R}_n^{+}(\mathbf{y}_i) = N_K^{-1} \sum_{\mathbf{k} \in K} \mathbf{Q}_{\mathbf{k}}^{+}(\mathbf{y}_i),$$

and the signed-rank statistic is

$$\mathbf{T} = \sum_{i=1}^{n} \mathbf{R}_n^+(\mathbf{y}_i).$$

Under the assumption that the observations are from a distribution that is symmetric around $\mathbf{0}$, the null hypothesis in (1.1) is rejected if $n^{-1}\mathbf{T}'\mathbf{B}^{-1}\mathbf{T} \geq \chi_p^2(\alpha)$, where $\mathbf{B} = n^{-1}\sum_{i=1}^{n} \mathbf{R}_n^+(\mathbf{y}_i)\mathbf{R}_n^{+'}(\mathbf{y}_i)$ and $\chi_p^2(\alpha)$ is the upper α quantile of a chi-square distribution with p degrees of freedom. Möttönen et al. (1998) gave a formula for calculating the Asymptotic Relative Efficiency (ARE) of this test statistic relative to Hotelling's T^2. Their results showed that the test statistic is more efficient than Hotelling's T^2 when the underlying distribution of the $\mathbf{Y}$'s is the multivariate t-distribution with small degrees of freedom and as the degrees of freedom increases, eventually to ∞, the test statistic performs very well as compared to Hotelling's T^2. A drawback of this test is that, as the dimension or the sample size increases, it becomes very difficult to calculate.

Peters and Randles (1990) and Jan and Randles (1994) developed an affine-invariant signed-rank test and signed sum test, respectively. Both procedures are modifications of Randles (1989) multivariate sign test based on the notion of interdirection. The interdirection c_{jk} of two observations $\mathbf{Y}_j$ and $\mathbf{Y}_k$ is defined as the number of hyper-planes formed by the origin and $p-1$ observations (excluding $\mathbf{Y}_j$ and $\mathbf{Y}_k$) such that $\mathbf{Y}_j$ and $\mathbf{Y}_k$ are on opposite sides of the hyperplane. Peters and Randles assumed that the observations come from an elliptically symmetric distribution and thus it was natural to rank the observations according to their distances from the origin measured in terms of elliptical contours. Using this idea of ranking, Peters and Randles proposed the statistic

$$W_n = \frac{3p}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n}\cos(\pi\hat{p}_{jk})\frac{R_j}{n}\frac{R_k}{n}, \tag{1.3}$$

where R_i denotes the rank of $\hat{D}_i = \mathbf{Y}_i'\hat{\mathbf{\Sigma}}^{-1}\mathbf{Y}_i$ among $\mathbf{Y}_1'\hat{\mathbf{\Sigma}}^{-1}\mathbf{Y}_1, \cdots, \mathbf{Y}_n'\hat{\mathbf{\Sigma}}^{-1}\mathbf{Y}_n$,

$$\hat{\mathbf{\Sigma}} = n^{-1}\sum_{i=1}^{n}\mathbf{Y}_i\mathbf{Y}_i',$$

$$\hat{p}_{jk} = \begin{cases} (c_{jk} + d_n)/\binom{n}{p-1} & \text{if } j \neq k \\ 0 & \text{if } j = k, \end{cases}$$

and $d_n = (1/2)\{\binom{n}{p-1} - \binom{n-2}{p-1}\}$. They proved that if the $\mathbf{Y}$s are iid from an elliptically symmetric density with var($\mathbf{Y}$)=$\mathbf{\Sigma}$ and $n^{1/2}(\hat{\mathbf{\Sigma}} - \mathbf{\Sigma}) \overset{H_o}{=} O_p(1)$, then W_n has a limiting χ_p^2 distribution under the null hypothesis. Peters and Randles showed through ARE calculations and a Monte Carlo study that W_n performs better than Hotelling's T^2 for both light-tailed and heavy-tailed distributions and virtually as well as Hotelling's T^2 under the multivariate normal distribution when $p = 2$ or 3. However, as the dimension increases, not only does W_n's computation become difficult but also its performance deteriorates, especially in the light-tailed cases and the normal case. In fact the Pitman efficiency of this test relative to Hotelling's T^2 goes to .75 as $p \to \infty$ for distribution in the power family (see Peters and Randles, 1990, p.553). The power family of distributions will be defined in more details in Chapter 3.

The signed sum test of Jan and Randles results from applying Randles (1989) sign test to the sums of pairs of observations. Thus, the statistic has the form

$$SS = \frac{4p}{n(n+1)^2}\sum_{i \leq k}\sum_{i' \leq k'}\cos(\pi\hat{p}_{ik,i'k'}), \tag{1.4}$$

where

$$\hat{p}_{ik,i'k'} = \begin{cases} \frac{C_{ik,i'k'} + d_n}{\binom{n}{p-1}} & \text{if } (i,k) \neq (i',k') \\ 0 & \text{if } (i,k) = (i',k'), \end{cases}$$

$$d_n = \frac{1}{2}\left[\binom{n}{p-1} - \binom{n-4}{p-1}\right],$$

and $C_{ik,i'k'}$ is the number of hyper-planes formed by the origin $\mathbf{0}$ and other $p-1$ observations (excluding $\mathbf{Y}_i, \mathbf{Y}_k, \mathbf{Y}_{i'}$, and $\mathbf{Y}_{k'}$) such that $\mathbf{Y}_i + \mathbf{Y}_k$ and $\mathbf{Y}_{i'} + \mathbf{Y}_{k'}$ are on opposite sides of the hyper-plane formed. This statistic has a limiting χ^2_p null distribution when the $\mathbf{Y}$s come from an elliptically symmetric distribution and a better efficiency than Hotelling's T^2 only under heavy-tailed distributions. While its Pitman efficiencies do not deteriorate as the dimension increases as was the case for the test by Peters and Randles, its calculation rapidly becomes quite difficult when the dimension and/or the sample size increases.

Hössjer and Croux (1995) developed a method to generalize signed-rank statistics to dimensions higher than one. When the underlying distribution of the $\mathbf{Y}$s is elliptically symmetric, they suggested transforming the observations using an equivariant estimate of the population covariance matrix, say $\hat{\mathbf{\Sigma}}_n$, that is based on the observations and then calculating a test statistic using the transformed observations. They based their test on the statistic

$$\mathbf{HC}_n = \sum_{i=1}^{n} a_n(R(\|\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_i\|))\frac{\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_i}{\|\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_i\|},$$

where, $a_n(1), \cdots, a_n(n)$ are non-negative scores satisfying some regularity conditions and $R(\|\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_i\|)$ is the rank of $\|\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_i\|$ among $\|\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_1\|, \cdots, \|\hat{\mathbf{\Sigma}}_n^{-1/2}\mathbf{Y}_n\|$. They suggested using $\hat{\mathbf{\Sigma}} = \mathbf{S}$ as an estimator of the variance-covariance matrix or a high break down estimator from the S-estimator class. In their paper, they also gave formulas for calculating the asymptotic relative efficiencies for these generalized tests with respect to Hotelling's T^2. They demonstrated that the efficiency depends to a large extent on the choice of the scores. For that matter, they showed that for the multivariate normal distribution the efficiency of the sign scores (i.e. $a_n \equiv 1$) approaches one as the dimension increases while the efficiency for the Wilcoxon scores (i.e. $a_n(i) = i$) decreases to .75. Note that, although Hössjer and Croux gave this generalization that is simple to apply, they did not suggest a "general" score function that can maintain a high efficiency regardless of the dimension. What is meant by general is that it works well for all elliptically symmetric distributions. In particular, their multivariate extension of the univariate signed-rank test ($a_n(i) = i$) has efficiencies that deteriorate to .75 (like those of Peters and Randles) for any distribution in the power family class.

Recently, Randles (2000) proposed a new sign test which has very desirable characteristics. It uses the transformation-retransformation approach of Chakraborty, Chaudhuri, and Oja (1998) together with a directional transformation due to Tyler (1987). This new multivariate sign test is practical to apply to data of any dimension, makes minimal assumptions about the underlying distribution and has a small sample distribution-free property over a broad class of distributions.

In Chapter 2, Randles (2000) sign test is described and, based on it, a new affine-invariant multivariate class of signed-rank tests is proposed. This new test makes minimal assumptions about the underlying population and is much easier to compute than any of the others in the literature (except possibly the test by Hössjer and Croux) for data in any practical dimension. It's affine-invariance property as well as its null limiting distribution are established. The asymptotic distribution of the new test, under Pitman alternatives, is investigated in Chapter 3, which also includes the comparisons of a particular member of this class with several one-sample multivariate procedures via asymptotic relative efficiencies and Monte-Carlo results. In particular, it is demonstrated that this new multivariate extension of the univariate signed-rank test has very desirable Pitman efficiencies and that these efficiencies do not deteriorate for $p \to \infty$, as was the case for the Peters-Randles and Hössjer-Croux extensions. The repeated measures problem setting is introduced in Chapter 4 along with a literature review of procedures used with such a design. The application of the new signed-rank test and Randles (2000) sign test to this problem is investigated in Chapter 5. Finally, several procedures for this setting are compared using asymptotic relative efficiencies and Monte-Carlo studies.

CHAPTER 2
A MULTIVARIATE ONE-SAMPLE TEST FOR LOCATION

2.1 Definition of the Test Statistic

Since the new signed-rank test statistic is a modification of Randles (2000) multivariate sign test, it is important to describe this sign test in more detail than the others described in Chapter 1.

Randles (2000) suggests using the transformation-retransformation approach where each observation $\mathbf{Y}_i$ is transformed by a data-determined nonsingular matrix $\hat{\mathbf{A}} \equiv \hat{\mathbf{A}}_{\mathbf{y}}$, to form $\hat{\mathbf{A}}\mathbf{Y}_i$. Unit vectors are created from the transformed data, via $\mathbf{V}_i = \frac{\hat{\mathbf{A}}\mathbf{Y}_i}{||\hat{\mathbf{A}}\mathbf{Y}_i||}$, where $||\cdot||$ is the Euclidean norm. The test statistic is then a quadratic form in the mean unit vector, i.e.,

$$S_n = n\overline{\mathbf{V}}'(n^{-1}\sum_{i=1}^{n}\mathbf{V}_i\mathbf{V}_i')^{-1}\overline{\mathbf{V}}. \tag{2.1}$$

In his paper, Randles chooses the matrix $\hat{\mathbf{A}} = \hat{\mathbf{A}}_d$ that was proposed by Tyler (1987), to satisfy:

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\hat{\mathbf{A}}_d\mathbf{Y}_i}{||\hat{\mathbf{A}}_d\mathbf{Y}_i||}\right)\left(\frac{\hat{\mathbf{A}}_d\mathbf{Y}_i}{||\hat{\mathbf{A}}_d\mathbf{Y}_i||}\right)' = \frac{1}{p}\mathbf{I}. \tag{2.2}$$

Beside being nonsingular, $\hat{\mathbf{A}}_d$ is also invariant under sign changes among the $\mathbf{y}_i$'s. Moreover, $\hat{\mathbf{A}}_d$ satisfies the following affine-equivariance property:

$$\mathbf{D}'\hat{\mathbf{A}}_{\mathbf{Dy}}'\hat{\mathbf{A}}_{\mathbf{Dy}}\mathbf{D} = c_o\hat{\mathbf{A}}_{\mathbf{y}}'\hat{\mathbf{A}}_{\mathbf{y}}, \tag{2.3}$$

where $\mathbf{D}$ is a fixed nonsingular $p \times p$ matrix, $\hat{\mathbf{A}}_{\mathbf{Dy}}$ is the $\hat{\mathbf{A}}$ calculated on the transformed observations $\mathbf{Dy}_i$, and c_o is a positive scalar that may depend on $\mathbf{D}$ and the $\mathbf{y}_i$'s. Equation (2.2) shows that the unit vectors of the transformed data have a variance-covariance structure like that of a random variable that is uniform on the p-unit sphere. Tyler (1987) not only showed that $\hat{\mathbf{A}}_d$ is unique if the sample is drawn from a continuous p-dimensional distribution and $n > p(p-1)$ but also proved that $\hat{\mathbf{A}}_d$ is consistent and asymptotically normal. In Appendix A, an iterative scheme for computing $\hat{\mathbf{A}}_d$ is described, it was given by Tyler and it proved to compute fast for data in any practical dimension using OX, the object oriented language. With this choice of $\hat{\mathbf{A}}$, (2.1) becomes

$$S_n = np\overline{\mathbf{V}}'\overline{\mathbf{V}} \tag{2.4}$$

and H_o is rejected for large values. The test statistic S_n is affine-invariant. Under H_o, S_n is distribution-free for the class of distributions with elliptical directions and conditionally distribution-free for the broader class of directionally symmetric distributions. Randles (2000) proved that if the underlying distribution is directionally symmetric around $\mathbf{0}$, then $S_n \xrightarrow{d} \chi^2_p$.

We now propose a class of signed-rank statistics related to (2.1) and (2.4). These statistics use the ranks of the distances of the transformed observations from the origin. Specifically, let Q_i be the rank of $\|\hat{\mathbf{A}}_d\mathbf{Y}_i\|$ among $\|\hat{\mathbf{A}}_d\mathbf{Y}_1\|, \cdots, \|\hat{\mathbf{A}}_d\mathbf{Y}_n\|$ and define

$$\mathbf{V}_{si} = \phi\left(\frac{Q_i}{n+1}\right)\mathbf{V}_i,$$

where $\phi(\cdot)$ is a nonnegative, nondecreasing, uniformly bounded continuous function that may depend on the dimension p. The test statistic is then the quadratic form

$$W_{np} = n\bar{\mathbf{V}}_s'(n^{-1}\sum_{i=1}^{n}\mathbf{V}_{si}\mathbf{V}_{si}')^{-1}\bar{\mathbf{V}}_s. \tag{2.5}$$

and H_o is rejected for large values of W_{np}. The test statistic W_{np} is affine-invariant as long as $n > p(p-1)$ (see next section). Moreover, if under H_o the $\mathbf{Y}_i$'s are iid from a symmetric density (i.e. $\mathbf{Y}_i \overset{d}{=} -\mathbf{Y}_i$), then W_{np} is conditionally distribution-free (conditioning on the observed $\pm\mathbf{y}_i$'s). Using the fact that $\hat{\mathbf{A}}_{\mathbf{d}}$, and thus the

Q_i's, are invariant under changes in the signs among the $\mathbf{y}_i$'s, the conditional null distribution of W_{np} is easy to generate. If we let $\delta_i = \pm 1$ for $i = 1, \cdots, n$, then these 2^n equally likely choices for the δ-vector yield 2^n possible conditional values of the test statistic via:

$$\begin{aligned} W_{np\delta} &= n \left(\frac{1}{n} \sum_{i=1}^{n} \phi\left(\frac{q_i}{n+1} \right) \delta_i \mathbf{v}_i \right)' \left(n^{-1} \sum_{i=1}^{n} \left[\phi\left(\frac{q_i}{n+1} \right) \right]^2 \mathbf{v}_i \mathbf{v}_i' \right)^{-1} \\ &\quad \left(\frac{1}{n} \sum_{i=1}^{n} \phi\left(\frac{q_i}{n+1} \right) \delta_i \mathbf{v}_i \right) \end{aligned}$$

where q_i and $\mathbf{v}_i$ are the observed values of Q_i and $\mathbf{V}_i$, respectively $i = 1, \cdots, n$. Finally, H_o is rejected at level α if the observed W_{np} value is among the largest $100 \cdot \alpha$ percent of the 2^n equally likely values of $W_{np\delta}$.

2.2 The Choice of the ϕ Function

The $\phi(\cdot)$ function plays an important role in determining the Pitman Asymptotic Relative Efficiency (ARE) of W_{np}. If $\phi(u) = 1$, for all u, then W_{np} reduces to Randles (2000) sign test. In such a case, and for elliptically symmetric distributions, W_{np} will have the same ARE as the tests proposed by Randles (1989) and Hettmansperger et al. (1994). Recall that these tests have higher efficiencies than Hotelling's T^2 for heavy-tailed distributions but do not perform as well for light-tailed distributions. Now if $\phi(u) = u$, for all u, then, for elliptically symmetric distributions, W_{np} will have the same ARE as Peters and Randles (1990) signed-rank test which performs better than Hotelling's T^2 for both light-tailed and heavy-tailed distributions and virtually as well as Hotteling's T^2 under the multivariate normal distribution when $p = 2$ or 3. However, as the dimension increases, not only does the test of Peters and Randles become computationally difficult but also its performance deteriorates, especially in the light-tailed cases and the normal case. In fact the Pitman efficiency of this test relative to Hotelling's T^2 goes to .75 as $p \to \infty$ for any distribution in the power family class.

In this dissertation, the function

$$\phi(u) = \lambda u + (1 - \lambda)1, \tag{2.6}$$

is recommended. Here, $\lambda = \{\ln(p + e - 1)\}^{-1}$ depends on p, the dimension of the data. When using this $\phi(\cdot)$ function, the test statistic will be denoted by W_{lnp}. This choice is motivated first by the simplicity of a linear function and second by the fact that when $p = 1$, $\phi(u) = u$ and the test is the two-tailed Wilcoxon signed rank test. But the most important motivation for this choice is the excellent asymptotic relative efficiencies of W_{lnp} established in Chapter 3. When the underlying distribution is the multivariate normal, for example, the tests by Peters and Randles and by Hössjer and Croux have ARE's, when compared to Hotelling's T^2, that approach .75 as $p \to \infty$, while the ARE of W_{lnp} approaches 1 as $p \to \infty$. Moreover, the ARE's of W_{lnp} are strong across a wide spectrum of distributions from very light-tailed to very heavy-tailed population models.

2.3 Affine-Invariance of W_{np}

In this section the affine-invariance property of W_{np} is established. This is done by proving the following theorem.

Theorem 2.3.1. *If* $\mathbf{D}$ *is a* $p \times p$ *nonsingular matrix, then*

$$W_{np}(\mathbf{Y}_1, \cdots, \mathbf{Y}_n) = W_{np}(\mathbf{D}\mathbf{Y}_1, \cdots, \mathbf{D}\mathbf{Y}_n),$$

where $W_{np}(\mathbf{Y}_1, \cdots, \mathbf{Y}_n)$ *denotes the proposed statistic,* W_{np}*, applied to the observations* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$.

Proof. Following the notation in Randles (2000), let $\hat{\mathbf{A}}_{\mathbf{y}d}$ and $\hat{\mathbf{A}}_{\mathbf{x}d}$ denote $\hat{\mathbf{A}}_d$ computed on the $\mathbf{y}_i$'s and $\mathbf{x}_i$'s respectively, where $\mathbf{x}_i = \mathbf{D}\mathbf{y}_i$ for $i = 1, \cdots, n$. By the affine-equivariance property (2.3) of $\hat{\mathbf{A}}_d$,

$$\mathbf{D}'\hat{\mathbf{A}}'_{\mathbf{x}d}\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{D} = c_o\hat{\mathbf{A}}'_{\mathbf{y}d}\hat{\mathbf{A}}_{\mathbf{y}d},$$

where c_o is a positive scalar. Therefore, for $i = 1, \cdots, n$ we have

$$\begin{aligned} \left\|\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_i\right\| &= \left(\mathbf{x}_i'\hat{\mathbf{A}}_{\mathbf{x}d}'\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_i\right)^{\frac{1}{2}} \\ &= \left(\mathbf{y}_i'\mathbf{D}'\hat{\mathbf{A}}_{\mathbf{x}d}'\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{D}\mathbf{y}_i\right)^{\frac{1}{2}} \\ &= c_o^{\frac{1}{2}}\left(\mathbf{y}_i'\hat{\mathbf{A}}_{\mathbf{y}d}'\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_i\right)^{\frac{1}{2}} \\ &= c_o^{\frac{1}{2}}\left\|\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_i\right\| \end{aligned}$$

The above equality along with the fact that c_o is positive implies that the rank of $\left\|\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_i\right\|$ among $\left\|\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_1\right\|, \cdots, \left\|\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_n\right\|$ is equal to the rank $\left\|\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_i\right\|$ among $\left\|\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_1\right\|, \cdots, \left\|\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_n\right\|, i = 1, \cdots, n$. Moreover, Randles (2000) proved that

$$\mathbf{V}_{i\mathbf{x}} \equiv \left(\frac{\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_i}{\left\|\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{x}_i\right\|}\right) = \boldsymbol{\Theta}\left(\frac{\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_i}{\left\|\hat{\mathbf{A}}_{\mathbf{y}d}\mathbf{y}_i\right\|}\right) \equiv \boldsymbol{\Theta}\mathbf{V}_{i\mathbf{y}},$$

where $\boldsymbol{\Theta} = c_o^{-\frac{1}{2}}\hat{\mathbf{A}}_{\mathbf{x}d}\mathbf{D}\hat{\mathbf{A}}_{\mathbf{y}d}^{-1}$ is an orthogonal $p \times p$ matrix. Thus, it is true that

$$\mathbf{V}_{si\mathbf{x}} \equiv \phi\left(\frac{Q_{i\mathbf{x}}}{n+1}\right)\mathbf{V}_{i\mathbf{x}} = \boldsymbol{\Theta}\phi\left(\frac{Q_{i\mathbf{y}}}{n+1}\right)\mathbf{V}_{i\mathbf{y}} \equiv \boldsymbol{\Theta}\mathbf{V}_{si\mathbf{y}}.$$

Therefore,

$$\begin{aligned} W_{np}(\mathbf{D}\mathbf{Y}_1, \cdots, \mathbf{D}\mathbf{Y}_n) &= W_{np}(\mathbf{X}_1, \cdots, \mathbf{X}_n) \\ &= n\bar{\mathbf{V}}_{s\mathbf{x}}'(n^{-1}\sum_{i=1}^{n}\mathbf{V}_{si\mathbf{x}}\mathbf{V}_{si\mathbf{x}}')^{-1}\bar{\mathbf{V}}_{s\mathbf{x}} \\ &= n\bar{\mathbf{V}}_{s\mathbf{y}}'\boldsymbol{\Theta}'(n^{-1}\sum_{i=1}^{n}\boldsymbol{\Theta}\mathbf{V}_{si\mathbf{y}}\mathbf{V}_{si\mathbf{y}}'\boldsymbol{\Theta}')^{-1}\boldsymbol{\Theta}\bar{\mathbf{V}}_{s\mathbf{y}} \\ &= n\bar{\mathbf{V}}_{s\mathbf{y}}'\boldsymbol{\Theta}'\boldsymbol{\Theta}'^{-1}(n^{-1}\sum_{i=1}^{n}\mathbf{V}_{si\mathbf{y}}\mathbf{V}_{si\mathbf{y}}')^{-1}\boldsymbol{\Theta}^{-1}\boldsymbol{\Theta}\bar{\mathbf{V}}_{s\mathbf{y}} \\ &= n\bar{\mathbf{V}}_{s\mathbf{y}}'(n^{-1}\sum_{i=1}^{n}\mathbf{V}_{si\mathbf{y}}\mathbf{V}_{si\mathbf{y}}')^{-1}\bar{\mathbf{V}}_{s\mathbf{y}} \\ &= W_{np}(\mathbf{Y}_1, \cdots, \mathbf{Y}_n). \end{aligned}$$

This completes the proof. □

2.4 The Asymptotic Null Distribution of W_{np}

In this section, the asymptotic null distribution of W_{np} under the class of symmetric distributions, is investigated. To do so, some notation is introduced. For $i = 1, \cdots, n$, let

$$\mathbf{V}_i = \frac{\hat{\mathbf{A}}_d \mathbf{Y}_i}{\|\hat{\mathbf{A}}_d \mathbf{Y}_i\|} \qquad \mathbf{V}_{si} = \phi\left(\frac{Q_i}{n+1}\right)\mathbf{V}_i$$
$$\mathbf{V}_{io} = \frac{\mathbf{A}_o \mathbf{Y}_i}{\|\mathbf{A}_o \mathbf{Y}_i\|} \qquad \mathbf{V}_{sio} = \phi\left(G(R_i)\right)\mathbf{V}_{io}$$

where $R_i = \mathbf{Y}_i'\mathbf{A}_o'\mathbf{A}_o\mathbf{Y}_i$, $G(t) = P_{H_o}(R_i \leq t)$, $(t \geq 0)$, and $\mathbf{A}_o$ is the unique solution among upper triangular matrices with 1 in the upper-left element satisfying

$$E\left[\left(\frac{\mathbf{A}_o\mathbf{Y}_i}{\|\mathbf{A}_o\mathbf{Y}_i\|}\right)\left(\frac{\mathbf{A}_o\mathbf{Y}_i}{\|\mathbf{A}_o\mathbf{Y}_i\|}\right)'\right] = \frac{1}{p}\mathbf{I}. \tag{2.7}$$

In section 3 of his paper, Tyler (1987) showed that

$$\sqrt{n}\left(\hat{\mathbf{A}}_d - \mathbf{A}_o\right) = \mathbf{O}_p(1). \tag{2.8}$$

Now, consider $\bar{\mathbf{V}}_s = \frac{1}{n}\sum_{i=1}^n \mathbf{V}_{si}$ and its approximating statistic $\bar{\mathbf{V}}_{so} = \frac{1}{n}\sum_{i=1}^n \mathbf{V}_{sio}$. In order to establish the asymptotic null distribution, the following lemma is needed.

Lemma 2.4.1. *Under H_o, if $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid from a distribution that is symmetric around $\mathbf{0}$, then*

$$\sqrt{n}\left(\bar{\mathbf{V}}_s - \bar{\mathbf{V}}_{so}\right) = \mathbf{o}_p(1). \tag{2.9}$$

and

$$\left[\frac{1}{n}\sum_{i=1}^n \mathbf{V}_{si}\mathbf{V}_{si}' - \frac{1}{n}\sum_{i=1}^n \mathbf{V}_{sio}\mathbf{V}_{sio}'\right] = \mathbf{o}_p(1). \tag{2.10}$$

Proof. We start the proof by showing that under H_o and for all i,

$$G_n\left(\mathbf{Y}_i'\hat{\mathbf{A}}_d'\hat{\mathbf{A}}_d\mathbf{Y}_i\right) \xrightarrow{p} G\left(\mathbf{Y}_i'\mathbf{A}_o'\mathbf{A}_o\mathbf{Y}_i\right), \tag{2.11}$$

where $G_n(t) = \frac{1}{n}\sum_{i=1}^{n} I\left(\|\hat{\mathbf{A}}_d\mathbf{Y}_i\|^2 \le t\right)$, so $G_n\left(\mathbf{Y}_i'\hat{\mathbf{A}}_d'\hat{\mathbf{A}}_d\mathbf{Y}_i\right) = Q_i/n$. Applying lemma 2 of Peters and Randles (1990) to (2.11), it is seen that we need to prove the two conditions:

$$\sqrt{n}\left(\hat{\mathbf{A}}_d'\hat{\mathbf{A}}_d - \mathbf{A}_o'\mathbf{A}_o\right) = \mathbf{O}_p(1) \tag{2.12}$$

$$\sup_{\mathbf{s}\in\boldsymbol{\beta}} \left|\mathbf{Y}_i'(\mathbf{A}_o'\mathbf{A}_o + n^{-1/2}\mathbf{s})\mathbf{Y}_i - \mathbf{Y}_i'\mathbf{A}_o'\mathbf{A}_o\mathbf{Y}_i\right| = \mathbf{o}_p(1), \tag{2.13}$$

where $\boldsymbol{\beta}$ is the set of all $p \times p$ matrices in a bounded sphere centered at $\mathbf{0}_{p\times p}$. Tyler (1987) essentially proved (2.12) (see theorem 3.2 on page 243). To verify (2.13), let $M > 0$ be the bound on each element of the matrices in $\boldsymbol{\beta}$, then for $\mathbf{s} \in \boldsymbol{\beta}$ we have

$$\begin{aligned}
\left|\mathbf{Y}_i'(\mathbf{A}_o'\mathbf{A}_o + n^{-1/2}\mathbf{s})\mathbf{Y}_i - \mathbf{Y}_i'\mathbf{A}_o'\mathbf{A}_o\mathbf{Y}_i\right| &= \left|\mathbf{Y}_i' n^{-1/2}\mathbf{s}\mathbf{Y}_i\right| \\
&= \left| n^{-1/2}\sum_{t=1}^{p}\sum_{t'=1}^{p} Y_{it}Y_{it'}s_{tt'}\right| \\
&\le n^{-1/2}M\sum_{t=1}^{p}\sum_{t'=1}^{p}\left|Y_{it}Y_{it'}\right|,
\end{aligned}$$

where $s_{tt'}$ is the $(t,t')^{th}$ element of $\mathbf{s}$. Therefore

$$\sup_{\mathbf{s}\in\boldsymbol{\beta}} \left|\mathbf{Y}_i'(\mathbf{A}_o'\mathbf{A}_o + n^{-1/2}\mathbf{s})\mathbf{Y}_i - \mathbf{Y}_i'\mathbf{A}_o'\mathbf{A}_o\mathbf{Y}_i\right| \le n^{-1/2}M\sum_{t=1}^{p}\sum_{t'=1}^{p}\left|\mathbf{Y}_{it}\mathbf{Y}_{it'}\right| \xrightarrow{p} 0.$$

Thus (2.13) is established and hence (2.11) also follows.

Now to prove (2.9), write $\sqrt{n}\left(\bar{\mathbf{V}}_s - \bar{\mathbf{V}}_{so}\right)$ as

$$\begin{aligned}
\sqrt{n}\left[\frac{1}{n}\sum_{i=1}^{n}\phi\left(\frac{Q_i}{n+1}\right)\mathbf{V}_i - \frac{1}{n}\sum_{i=1}^{n}\phi\left(G(R_i)\right)\mathbf{V}_{io}\right] = & \\
\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\phi\left(\frac{Q_i}{n+1}\right)\left[\mathbf{V}_i - \mathbf{V}_{io}\right]\right\} & \qquad (2.14) \\
+\sqrt{n}\left\{\frac{1}{n}\sum_{i=1}^{n}\left[\phi\left(\frac{Q_i}{n+1}\right) - \phi\left(G(R_i)\right)\right]\mathbf{V}_{io}\right\}. & \qquad (2.15)
\end{aligned}$$

The k^{th} element of the vector in (2.14) is

$$c_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \phi\left(\frac{Q_i}{n+1}\right) \left[\frac{\hat{\mathbf{a}}_k' \mathbf{Y}_i}{||\hat{\mathbf{A}}_d \mathbf{Y}_i||} - \frac{\mathbf{a}_{ko}' \mathbf{Y}_i}{||\mathbf{A}_o \mathbf{Y}_i||}\right],$$

where $\hat{\mathbf{a}}_k'$ and $\mathbf{a}_{ko}'$ are the k^{th} rows of $\hat{\mathbf{A}}_d$ and $\mathbf{A}_o$ respectively. Let $f(\mathbf{A}) = \frac{\mathbf{a}_k' \mathbf{Y}_i}{||\mathbf{A}\mathbf{Y}_i||}$, then the derivative of the univariate function $f(\cdot)$ with respect to the j^{th} row of $\mathbf{A}$ is

$$\mathbf{B}_k(j, \mathbf{A}, \mathbf{Y}) = \begin{cases} -\frac{(\mathbf{a}_k'\mathbf{Y})(\mathbf{a}_j'\mathbf{Y})}{||\mathbf{A}\mathbf{Y}||^3}\mathbf{Y} & \text{if } k \neq j, \\ \frac{\mathbf{Y}}{||\mathbf{A}\mathbf{Y}||} - \frac{(\mathbf{a}_j'\mathbf{Y})^2\mathbf{Y}}{||\mathbf{A}\mathbf{Y}||^3} & \text{if } k = j. \end{cases} \tag{2.16}$$

A Taylor series expansion of c_k yields

$$\sum_{j=1}^{p} \left[\sqrt{n}(\hat{\mathbf{a}}_j - \mathbf{a}_{jo})' \frac{1}{n} \sum_{i=1}^{n} \phi\left(\frac{Q_i}{n+1}\right) \mathbf{B}_k(j, \tilde{\mathbf{A}}_{ik}, \mathbf{Y}_i)\right],$$

where $\tilde{\mathbf{A}}_{ik}$ is between $\hat{\mathbf{A}}_{\mathbf{d}}$ and $\mathbf{A}_o$. Conditioning on all $\mathbf{Y}_j, j \neq i$,

$$\phi\left(\frac{Q_i}{n+1}\right) \mathbf{B}_k(j, \tilde{\mathbf{A}}_{ik}, \mathbf{Y}_i) \quad \text{and} \quad \phi\left(\frac{Q_i}{n+1}\right) \mathbf{B}_k(j, \tilde{\mathbf{A}}_{ik}, \mathbf{Y}_i) \phi\left(\frac{Q_{i'}}{n+1}\right) \mathbf{B}_k(j, \tilde{\mathbf{A}}_{i'k}, \mathbf{Y}_{i'})$$

are odd axial step functions in $\mathbf{Y}_i$ and thus, by lemma (A.2) of Randles (2000), their conditional, and thus unconditional, expectation is zero. Hence, by Markov's theorem $\frac{1}{n}\sum_{i=1}^{n} \phi\left(\frac{Q_i}{n+1}\right) \mathbf{B}_k(j, \tilde{\mathbf{A}}_{ik}, \mathbf{Y}_i) \xrightarrow{p} 0$ (see Gnedenko, 1962, p. 232). Moreover, $\sqrt{n}(\hat{\mathbf{a}}_j - \mathbf{a}_{jo})$ is bounded in probability; hence,

$$c_k \xrightarrow{p} 0.$$

This proves that (2.14) goes in probability to zero. Next, we show that (2.15) also goes in probability to zero. To do so, consider the k^{th} element of that vector, say,

$$l_k = \sqrt{n} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left[\phi\left(\frac{Q_i}{n+1}\right) - \phi\left(G(R_i)\right)\right] \frac{\mathbf{a}_{ok}' \mathbf{Y}_i}{||\mathbf{A}_o \mathbf{Y}_i||} \right\}$$

and let

$$r_{ik} = \left[\phi\left(\frac{Q_i}{n+1}\right) - \phi\left(G(R_i)\right)\right] \frac{\mathbf{a}_{ok}' \mathbf{Y}_i}{||\mathbf{A}_o \mathbf{Y}_i||}.$$

Then,

$$\begin{aligned} E\left[l_k^2\right] &= \frac{1}{n}\left[\left(\sum_{i=1}^{n} r_{ik}\right)^2\right] \\ &= \frac{1}{n}E\left[\sum_{i=1}^{n} r_{ik}^2 + 2\sum_{i<j} r_{ik}r_{jk}\right] \\ &= \frac{1}{n}E\left[\sum_{i=1}^{n} r_{ik}^2\right] + \frac{2}{n}E\left[\sum_{i<j} r_{ik}r_{jk}\right] \\ &= E(r_{ik}^2) + \frac{2}{n}\binom{n}{2}E(r_{ik}r_{jk}). \end{aligned}$$

The last equality follows from the fact that the terms in each sum are exchangeable. Conditioning on $\mathbf{Y}_j$, $j \neq i$, $r_{ik}r_{jk}$ is an odd axial function in $\mathbf{Y}_i$, which implies that

$$E[r_{ik}r_{jk}] = 0.$$

Now consider

$$E[r_{ik}^2] = E\left[\left(\phi\left(\frac{Q_i}{n+1}\right) - \phi\left(G(R_i)\right)\right)^2 \left(\frac{\mathbf{a}_{ok}'\mathbf{Y}_i}{\|\mathbf{A}_o\mathbf{Y}_i\|}\right)^2\right].$$

Note that, $\frac{\mathbf{a}_{ok}'\mathbf{Y}_i}{\|\mathbf{A}_o\mathbf{Y}_i\|} = O_p(1)$ and $\left[\phi\left(\frac{Q_i}{n+1}\right) - \phi\left(G(R_i)\right)\right]^2 \xrightarrow{p} 0$ follows from (2.11) and the continuity of $\phi(\cdot)$. Therefore, $r_{ik}^2 \xrightarrow{p} 0$. Moreover, r_{ik} is uniformly bounded and thus, by the Lebesgue Dominated Convergence Theorem, $E[r_{ik}^2] \to 0$. This implies that $E[l_k^2] \to 0$. Applying Markov's inequality (see Billingsley, 1995, p.80) yields $l_k \xrightarrow{p} 0$. This completes the proof of (2.9).

To prove (2.10) we proceed as follows. The left side of (2.10) can be written as

$$\left[\frac{1}{n}\sum_{i=1}^{n}\mathbf{V}_{si}\mathbf{V}'_{si}-\frac{1}{n}\sum_{i=1}^{n}\mathbf{V}_{sio}\mathbf{V}'_{sio}\right]$$

$$\begin{aligned}
&= \left[\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\mathbf{V}_i\mathbf{V}'_i-\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(G(R_i)\right)\mathbf{V}_{io}\mathbf{V}'_{io}\right]\\
&= \frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\left[\mathbf{V}_i\mathbf{V}'_i-\mathbf{V}_{io}\mathbf{V}'_{io}\right] \qquad (2.17)\\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\left[\phi^2\left(\frac{Q_i}{n+1}\right)-\phi^2\left(G(R_i)\right)\right]\mathbf{V}_{io}\mathbf{V}'_{io}. \qquad (2.18)
\end{aligned}$$

Now, write (2.17) as

$$\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\left[(\mathbf{V}_i-\mathbf{V}_{io})(\mathbf{V}_i-\mathbf{V}_{io})'+\mathbf{V}_{io}(\mathbf{V}_i-\mathbf{V}_{io})'+(\mathbf{V}_i-\mathbf{V}_{io})\mathbf{V}'_{io}\right].$$

Using a Taylor series expansion, the $(s,t)^{th}$ element of this matrix is

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\left[\sum_{j=1}^{p}(\hat{\mathbf{a}}_j-\mathbf{a}_{jo})'\mathbf{B}_s(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i)\right]\left[\sum_{j'=1}^{p}(\hat{\mathbf{a}}_{j'}-\mathbf{a}_{j'o})'\mathbf{B}_t(j',\hat{\mathbf{A}}_i,\mathbf{Y}_i)\right]\\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)V_{ios}\left[\sum_{j=1}^{p}(\hat{\mathbf{a}}_j-\mathbf{a}_{jo})'\mathbf{B}_t(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i)\right]\\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\left[\sum_{j=1}^{p}(\hat{\mathbf{a}}_j-\mathbf{a}_{jo})'\mathbf{B}_s(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i)\right]V_{iot}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\left[\sum_{j=1}^{p}\sum_{k=1}^{j}\left(\hat{\mathbf{a}}_{jk}-\mathbf{a}_{jko}\right)B_{sk}\left(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i\right)\right]\\
&\qquad\left[\sum_{j'=1}^{p}\sum_{k'=1}^{j'}\left(\hat{\mathbf{a}}_{j'k'}-\mathbf{a}_{j'k'o}\right)B_{tk'}\left(j',\hat{\mathbf{A}}_i,\mathbf{Y}_i\right)\right]\\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)V_{ios}\left[\sum_{j=1}^{p}\sum_{k=1}^{j}\left(\hat{\mathbf{a}}_{jk}-\mathbf{a}_{jko}\right)B_{tk}\left(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i\right)\right]\\
&\quad +\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right)\left[\sum_{j=1}^{p}\sum_{k=1}^{j}\left(\hat{\mathbf{a}}_{jk}-\mathbf{a}_{jko}\right)B_{sk}\left(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i\right)\right]V_{iot}
\end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{p}\sum_{k=1}^{j}\sum_{j'=1}^{p}\sum_{k'=1}^{j'} \left(\hat{\mathbf{a}}_{jk} - \mathbf{a}_{jko}\right)\left(\hat{\mathbf{a}}_{j'k'} - \mathbf{a}_{j'k'o}\right) \\
&\qquad \left\{\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right) B_{sk}\left(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i\right) B_{tk'}\left(j',\hat{\mathbf{A}}_i,\mathbf{Y}_i\right)\right\} \\
&\quad + \sum_{j=1}^{p}\sum_{k=1}^{j}\left(\hat{\mathbf{a}}_{jk} - \mathbf{a}_{jko}\right)\left\{\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right) V_{ios}B_{tk}\left(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i\right)\right\} \\
&\quad + \sum_{j=1}^{p}\sum_{k=1}^{j}\left(\hat{\mathbf{a}}_{jk} - \mathbf{a}_{jko}\right)\left\{\frac{1}{n}\sum_{i=1}^{n}\phi^2\left(\frac{Q_i}{n+1}\right) B_{sk}\left(j,\hat{\mathbf{A}}_i,\mathbf{Y}_i\right) V_{iot}\right\},
\end{aligned}$$

where $\mathbf{B}_s\left(j,\mathbf{A},\mathbf{Y}_i\right)$ is as defined in (2.16). The fact that $\mathbf{B}_k\left(j,\mathbf{A},\mathbf{Y}_i\right)$ is uniformly bounded for $\mathbf{A}$ in some neighborhood of $\mathbf{A}_o$, and $(\hat{\mathbf{A}} - \mathbf{A}_o) = \mathbf{o}_p(1)$ imply that the above summations all go in probability to zero. Now consider (2.18), an (s,t) element of that matrix is

$$w_{st} = \frac{1}{n}\sum_{i=1}^{n}\left[\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right] V_{ios}V_{iot}. \tag{2.19}$$

The scalars V_{ios} and V_{iot} are uniformly bounded and $\left[\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right] = o_p(1)$, thus the elements of the above summation are $o_p(1)$. Note also that the terms in (2.19) are exchangeable and uniformly bounded. Thus by the Lebesgue dominated convergence theorem (see Serfling, 1980, p. 11)

$$\lim_{n\to\infty} E\left[\left(\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right) V_{ios}V_{iot}\right] = 0.$$

Moreover,

$$\begin{aligned}
Var(w_{st}) &= E\left[\frac{1}{n}\sum_{i=1}^{n}\left(\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right) V_{ios}V_{iot}\right]^2 \\
&= \frac{1}{n^2}\left(\sum_{i=1}^{n} E\left[\left(\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right)^2 V_{ios}^2 V_{iot}^2\right]\right) \qquad (2.20) \\
&\quad + \frac{1}{n^2}\left(\sum_{i\neq j} E\left[\left(\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right)\right.\right. \\
&\qquad \left.\left.\left(\phi^2\left(\frac{Q_j}{n+1}\right) - \phi^2\left(G(R_j)\right)\right) V_{ios}V_{iot}V_{jos}V_{jot}\right]\right). \qquad (2.21)
\end{aligned}$$

Clearly (2.20) is $o_p(1)$, since the terms are exchangeable and the expected value is bounded. The terms in (2.21) are uniformly bounded and $o_p(1)$, thus,

$$\lim_{n\to\infty} E\left[\left(\phi^2\left(\frac{Q_i}{n+1}\right) - \phi^2\left(G(R_i)\right)\right)\left(\phi^2\left(\frac{Q_j}{n+1}\right) - \phi^2\left(G(R_j)\right)\right) V_{ios}V_{iot}V_{jos}V_{jot}\right] = 0.$$

An application of Markov's theorem implies that each element of the matrix (2.18) is $o_p(1)$, hence (2.18) is $\mathbf{o}_p(1)$. $\square$

With lemma (2.4.1) established, we are ready to find the limiting distribution of W_{np}. This is done by proving the following theorem:

Theorem 2.4.1. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from a distribution that is symmetric around* $\mathbf{0}$ *then, under* H_o, nW_{np} *converges in distribution to a chi-square distribution with* p *degrees of freedom.*

Proof. First, the limiting distribution of the approximate statistic $\sqrt{n}\bar{\mathbf{V}}_{so}$ is established. Consider $\bar{\mathbf{V}}_{so} = \frac{1}{n}\sum_{i=1}^{n} \phi\left(G(R_i)\right)\mathbf{V}_{io}$ and write $\mathbf{Y}_i = \delta_i \mathbf{Y}_i^*$. Here, $\delta_i = \text{sign}(Y_{i1})$, where Y_{i1} is the first component of the vector $\mathbf{Y}_i$, $\text{sign}(t) = 1, 0, -1$ as $t > 0, =, t < 0$, respectively, and $\mathbf{Y}_i^* = \delta_i \mathbf{Y}_i$. Under H_o, $\delta_1, \cdots, \delta_n$ are iid Bernoulli random variables with $p(\delta_i = 1) = p(\delta_i = -1) = \frac{1}{2}$. Moreover, $\delta_1, \cdots, \delta_n$ are independent of $\mathbf{Y}_1^*, \cdots, \mathbf{Y}_n^*$ and thus of $R_1 = \|\mathbf{A}_o\mathbf{Y}_1\|, \cdots, R_n = \|\mathbf{A}_o\mathbf{Y}_n\|$. Now consider the elements of the summation in $\bar{\mathbf{V}}_{so}$, the conditional expectation

$$\begin{aligned}
E\left[\phi\left(G(R_i)\right)\frac{\mathbf{A}_o\mathbf{Y}_i}{\|\mathbf{A}_o\mathbf{Y}_i\|}\,|\mathbf{Y}_i^*\right] &= E\left[\phi\left(G(R_i)\right)\delta_i\frac{\mathbf{A}_o\mathbf{Y}_i^*}{\|\mathbf{A}_o\mathbf{Y}_i^*\|}\,|\mathbf{Y}_i^*\right] \\
&= E\left[\phi\left(G(R_i)\right)\frac{\mathbf{A}_o\mathbf{Y}_i^*}{\|\mathbf{A}_o\mathbf{Y}_i^*\|}I_{[\delta_i=1]}\,|\mathbf{Y}_i^*\right] \\
&\quad -E\left[\phi\left(G(R_i)\right)\frac{\mathbf{A}_o\mathbf{Y}_i^*}{\|\mathbf{A}_o\mathbf{Y}_i^*\|}I_{[\delta_i=-1]}\,|\mathbf{Y}_i^*\right] \\
&= \mathbf{0}.
\end{aligned}$$

This implies that $E\left[\phi\left(G(R_i)\right)\frac{\mathbf{A}_o\mathbf{Y}_i}{||\mathbf{A}_o\mathbf{Y}_i||}\right] = \mathbf{0}$. Using a similar argument, we can prove that

$$E\left[\phi\left(G(R_i)\right)\phi\left(G(R_j)\right)\frac{\mathbf{A}_o\mathbf{Y}_i}{||\mathbf{A}_o\mathbf{Y}_i||}\left(\frac{\mathbf{A}_o\mathbf{Y}_j}{||\mathbf{A}_o\mathbf{Y}_j||}\right)'\right] = \mathbf{0}$$

To finish the proof, let $\mathbf{\Omega} = \text{Cov}\left(\phi\left(G(R_i)\right)\frac{\mathbf{A}_o\mathbf{Y}_i}{||\mathbf{A}_o\mathbf{Y}_i||}\right)$ then, by the Multivariate Central Limit Theorem,

$$\sqrt{n}\bar{\mathbf{V}}_{so} \overset{d}{\sim} N_p(\mathbf{0}, \mathbf{\Omega}). \tag{2.22}$$

This along with (2.9) imply that $\sqrt{n}\bar{\mathbf{V}}_s \overset{d}{\sim} N_p(\mathbf{0}, \mathbf{\Omega})$. Therefore, $n\bar{\mathbf{V}}_s'(\mathbf{\Omega})^{-1}\bar{\mathbf{V}}_s$ has a limiting χ^2_p null distribution. Finally, a simple application of Slutsky's theorem implies that W_{np} has a limiting χ^2_p null distribution. This is due to the fact that $\left[\frac{1}{n}\sum_{i=1}^n \mathbf{V}_{sio}\mathbf{V}_{sio}' - \mathbf{\Omega}\right] = o_p(1)$ and result (2.10). □

CHAPTER 3
TEST COMPARISONS

3.1 Asymptotic Distribution of W_{np} Under Contiguous Alternatives

In order to compute Pitman asymptotic relative efficiencies of W_{np}, the asymptotic distribution of W_{np} under contiguous alternatives is obtained. Thus, a sequence of alternatives approaching the null hypothesis is considered. In particular, consider

$$H_a : \boldsymbol{\theta} = \boldsymbol{\theta}_n = \frac{\mathbf{c}}{\sqrt{n}}, \tag{3.1}$$

where $\mathbf{c} = (c_1, \cdots, c_p)$ is a constant, yet arbitrary, vector. In this chapter, the asymptotic results are developed for the case when the $\mathbf{Y}_i$'s come from the class of elliptically symmetric distributions. Thus, each $\mathbf{Y}_i$ can be expressed as $\mathbf{Y}_i = R_i\mathbf{D}\mathbf{U}_i + \boldsymbol{\theta}$ where $\mathbf{D}$ is a fixed nonsingular $p \times p$ matrix, the $\mathbf{U}_i$'s are iid uniformly distributed on the unit p-sphere, and the R_i's are iid positive random variables independent of the $\mathbf{U}_i$'s (see Fang, Kotz, and Ng, 1990, book). Moreover, if $\mathbf{Y}$ is elliptically symmetric, then the density function of $\mathbf{Y}$ is of the form

$$f_{\boldsymbol{\theta},\boldsymbol{\Sigma}}(\mathbf{y}) = K_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}}h\{(\mathbf{y} - \boldsymbol{\theta})'\boldsymbol{\Sigma}^{-1}(\mathbf{y} - \boldsymbol{\theta})\}, \tag{3.2}$$

where $K_p > 0$, and $\boldsymbol{\Sigma}$ is proportional to the covariance matrix of $\mathbf{Y}$, provided it exits. Without loss of generality, take $\mathbf{D} = \mathbf{I}$, the $p \times p$ identity matrix which implies that $\boldsymbol{\Sigma} = \mathbf{I}$. This is valid because W_{np} is affine-invariant. Two specific families of elliptically symmetric distributions are considered. The first is called the power family which has the form in (3.2) with

$$h(t) = \exp\left\{-\left(\frac{t}{c_0}\right)^{\nu}\right\}, \qquad \nu > 0,$$

where

$$c_o = \frac{p\Gamma(p/2\nu)}{\Gamma((p+2)/2\nu)}, \qquad K_p = \frac{\nu\Gamma(p/2)}{\Gamma(p/2\nu)[\pi c_0]^{p/2}},$$

and

$$\Gamma(w) = \int_0^\infty u^{w-1}e^{-u}du, \qquad w > 0.$$

This family includes the multivariate normal density ($\nu = 1$), as well as heavier-tailed distributions ($0 < \nu < 1$) and lighter-tailed distributions ($\nu > 1$). The second is the multivariate t-distribution where $h(\cdot)$ and K_p in (3.2) are defined as

$$h(t) = \left[1 + \frac{1}{\nu}(t)\right]^{-(p+\nu)/2},$$

and

$$K_p = \frac{\Gamma((p+\nu)/2)}{\Gamma(\nu/2)(\pi\nu)^{p/2}}. \tag{3.3}$$

Before the asymptotic distribution of W_{np} is obtained, a few comments are essential. First, note that for elliptically symmetric distributions and under H_o, $\mathbf{A}_o$, the matrix defined in section 2.3 is such that $\mathbf{A}_o = \mathbf{I}$. This is easily seen from condition (2.7) and the uniqueness of $\mathbf{A}_o$. Second, the variance-covariance matrix $\mathbf{\Omega}$ defined in (2.22) is such that

$$\mathbf{\Omega} = kI_{p\times p}, \tag{3.4}$$

where,

$$k = \frac{1}{p}E_{H_o}\left[\phi^2(G(R_i))\right].$$

The next theorem describes the distribution of W_{np} under (3.1) for the general elliptically symmetric distribution.

Theorem 3.1.1. *If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid p-variate vectors from an elliptically symmetric distribution whose density is given in (3.2), then under the alternatives defined in (3.1), W_{np} has a limiting non-central chi-square distribution with p* *degrees of freedom and non-centrality parameter*

$$\delta_{W_{np}} = (\frac{4}{p})\frac{E_{H_o}^2\left[R\frac{h'(R^2)}{h(R^2)}\phi(G(R))\right]}{E_{H_o}^2[\phi^2(G(R))]}, \tag{3.5}$$

where

$$h'(R^2) = \frac{\partial h(t)}{\partial t}|_{t=R^2}.$$

Proof. Without loss of generality, consider the case $\mathbf{\Sigma} = \mathbf{I}$. Under H_o, the distribution of the $\mathbf{Y}$'s has the form

$$f(\mathbf{y}) = f_{\mathbf{0},\mathbf{I}}(\mathbf{y}) = K_p h(\mathbf{y}'\mathbf{y}).$$

Hence,

$$\partial_t f(\mathbf{y}) = K_p 2y_t h'(\mathbf{y}'\mathbf{y})$$

and

$$\frac{\partial_t f(\mathbf{y})}{f(\mathbf{y})} = 2y_t\frac{h'(\mathbf{y}'\mathbf{y})}{h(\mathbf{y}'\mathbf{y})}$$

Following the rationale in Chapter 7 of Hájek, Šidák and Sen (1999), let

$$\begin{aligned} T_n &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{c}'\left[\frac{\boldsymbol{\partial} f(\mathbf{y})}{f(\mathbf{y})}\right] \\ &= \frac{2\mathbf{c}'}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{y}_i\left[\frac{h'(\mathbf{y}_i'\mathbf{y}_i)}{h(\mathbf{y}_i'\mathbf{y}_i)}\right], \end{aligned}$$

where $\mathbf{c}$ is as defined in (3.1), and

$$L_n = \sqrt{n}\left(\lambda_1\frac{1}{n}\sum_{i=1}^{n}\phi(G(R_i))U_{i1},\cdots,\lambda_p\frac{1}{n}\sum_{i=1}^{n}\phi(G(R_i))U_{ip}\right).$$

Consider

$$\begin{aligned}
D_n &= a_1L_n + a_2T_n \\
&= \sqrt{n}a_1\lambda_1(\frac{1}{n}\sum_{i=1}^{n}\phi(G(R_i))U_{i1}) + \cdots + \sqrt{n}a_1\lambda_p(\frac{1}{n}\sum_{i=1}^{n}\phi(G(R_i))U_{ip}) \\
&\quad + \frac{2a_2}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{c}'\mathbf{y}_i\frac{h'(\mathbf{y}_i'\mathbf{y}_i)}{h(\mathbf{y}_i'\mathbf{y}_i)} \\
&= \sqrt{n}a_1\lambda_1(\frac{1}{n}\sum_{i=1}^{n}\phi(G(R_i))U_{i1}) + \cdots + \sqrt{n}a_1\lambda_p(\frac{1}{n}\sum_{i=1}^{n}\phi(G(R_i))U_{ip}) \\
&\quad + \frac{2a_2}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{c}'R_i\mathbf{U}_i\frac{h'(R_i^2)}{h(R_i^2)} \\
&= (1,1,\cdots,1)\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\begin{bmatrix} \left(2a_2c_1R_i\frac{h'(R_i^2)}{h(R_i^2)} + a_1\lambda_1\phi(G(R_i))\right)U_{i1} \\ \vdots \\ \left(2a_2c_pR_i\frac{h'(R_i^2)}{h(R_i^2)} + a_1\lambda_p\phi(G(R_i))\right)U_{ip}\end{bmatrix}.
\end{aligned}$$

Each term on the right has zero expectation because the R_i and U_{it} are independent and $E_{H_o}[U_{it}] = 0$, for all $1 \leq i \leq n$ and all $1 \leq t \leq p$. Moreover, the terms in the bracket are uncorrelated because $E_{H_o}[U_{it}U_{it'}] = 0$, for all $t \neq t'$. Now consider

$$\begin{aligned}
\sigma_{tt} &= E_{H_o}\left[\left(2a_2c_tR_i\frac{h'(R_i^2)}{h(R_i^2)} + a_1\lambda_1\phi(G(R_i))\right)^2 U_{it}^2\right] \\
&= E_{H_o}\left[U_{it}^2\right]\left\{(2a_2c_t)^2 E_{H_o}\left[R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]^2\right. \\
&\quad \left. + 2\,(2a_2c_t)(a_1\lambda_t)E_{H_o}\left[\phi(G(R_i))R_i\frac{h'(R_i^2)}{h(R_i^2)}\right] + a_1^2\lambda_t^2E_{H_o}\left[\phi^2(G(R_i))\right]\right\} \\
&= \frac{1}{p}\left\{(2a_2c_t)^2 E_{H_o}\left[R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]^2\right. \\
&\quad \left. + 2\,(2a_2c_t)(a_1\lambda_t)E_{H_o}\left[\phi(G(R_i))R_i\frac{h'(R_i^2)}{h(R_i^2)}\right] + a_1^2\lambda_t^2E_{H_o}\left[\phi^2(G(R_i))\right]\right\}.
\end{aligned}$$

Therefore, by the multivariate central limit theorem

$$D_n \xrightarrow{d} N(0, \sigma_{11} + \cdots + \sigma_{pp}),$$

where

$$\begin{aligned}\sigma_{11}+\cdots+\sigma_{pp} &= \frac{1}{p}(2a_2)^2 E_{H_o}\left[R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]^2\sum_{t=1}^{p}c_t^2 \\ &\quad + 2\frac{2a_1a_2}{p}E_{H_o}\left[\phi(G(R_i))R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]\sum_{t=1}^{p}c_t\lambda_t \\ &\quad + \frac{a_1^2}{p}E_{H_o}\left[\phi^2(G(R_i))\right]\sum_{t=1}^{p}\lambda_t^2.\end{aligned}$$

Note that $L_n = \sqrt{n}\boldsymbol{\lambda}'\bar{\mathbf{V}}_{so}$, where $\boldsymbol{\lambda} = (\lambda_1,\cdots,\lambda_p)$. Thus, using (2.22) and (3.4), it is clear that

$$L_n \xrightarrow{d} N(0, \frac{1}{p}E_{H_o}\left[\phi^2(G(R_i))\right]\sum_{t=1}^{p}\lambda_t^2). \tag{3.6}$$

Moreover, T_n can be written as $T_n = \frac{2}{\sqrt{n}}\sum_{i=1}^{n}\left[R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]\mathbf{c}'\mathbf{U}_i$, thus

$$E_{H_o}[T_n] = 0,$$

and,

$$\text{Var}(T_n) = \sigma_T^2 = \frac{4}{p}E_{H_o}\left[R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]^2\sum_{t=1}^{p}c_t^2.$$

Moreover, $I_{\mathbf{c}}(f) = \sigma_T$, where $I_{\mathbf{c}}(f)$ represents the Fisher information of $f(\cdot)$. Therefore it follows from Le Cam's third lemma that, under the alternatives in (3.1),

$$L_n \xrightarrow{d} N\left(\frac{2}{p}E_{H_o}\left[\phi(G(R_i))R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]\mathbf{c}'\boldsymbol{\lambda}, \frac{1}{p}E_{H_o}\left[\phi^2(G(R_i))\right]\boldsymbol{\lambda}'\boldsymbol{\lambda}\right),$$

and thus

$$\sqrt{n}\bar{\mathbf{V}}_{so} \xrightarrow{d} N\left(\frac{2}{p}E_{H_o}\left[\phi(G(R_i))R_i\frac{h'(R_i^2)}{h(R_i^2)}\right]\mathbf{c}, \frac{1}{p}E_{H_o}\left[\phi^2(G(R_i))\right]\mathbf{I}\right).$$

Therefore, under the contiguous alternatives in (3.1),

$$W_{np} \xrightarrow{d} \chi_p^2\left((\frac{4}{p})\frac{E_{H_o}^2\left[R_i\frac{h'(R_i^2)}{h(R_i^2)}\phi(G(R_i))\right]}{E_{H_o}^2[\phi^2(G(R_i))]}\mathbf{c}'\mathbf{c}\right). \tag{3.7}$$

□

Corollary 3.1.1. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from the power family of distributions, then under the alternatives defined in (3.1),* W_{np} *has a non-central chi-square distribution with p degrees of freedom and non-centrality parameter*

$$\delta_{1W} = \left(\left(\frac{2\nu}{\sqrt{p} c_o^{\nu}} \right)^2 \frac{E_{H_o}^2 \left[\phi(G(R)) R^{2\nu-1} \right]}{E_{H_o} \left[\phi^2(G(R)) \right]} \mathbf{c}'\mathbf{c} \right). \tag{3.8}$$

Proof. The proof follows from theorem 3.1.1 and the fact that for the power family

$$h(t) = \exp\{-(\frac{t}{c_o})^{\nu}\},$$

and

$$\begin{aligned} \frac{h'(t)}{h(t)} &= \frac{-\frac{\nu t^{\nu-1}}{c_o^{\nu}} \exp\{-(\frac{t}{c_o})^{\nu}\}}{\exp\{-(\frac{t}{c_o})^{\nu}\}} \\ &= -\frac{\nu t^{\nu-1}}{c_o^{\nu}}. \end{aligned}$$

□

Corollary 3.1.2. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from the multivariate t-distribution family, then under the alternatives defined in (3.1),* W_{np} *has a non-central chi-square distribution with p degrees of freedom and non-centrality parameter*

$$\delta_{2W} = \left(\left(\frac{p+\nu}{\sqrt{p}\nu} \right)^2 \frac{E_{H_o}^2 \left[\phi(G(R)) R / (1 + \frac{1}{\nu} R^2) \right]}{E_{H_o} \left[\phi^2(G(R)) \right]} \mathbf{c}'\mathbf{c} \right), \tag{3.9}$$

Proof. The multivariate t-distribution is such that

$$h(t) = \left[1 + \frac{t}{\nu} \right]^{-\frac{p+\nu}{2}},$$

and

$$\begin{aligned} \frac{h'(t)}{h(t)} &= \frac{(-\frac{p+\nu}{2})(\frac{1}{\nu}) \left[1 + \frac{t}{\nu} \right]^{-\frac{p+\nu}{2} - 1}}{\left[1 + \frac{t}{\nu} \right]^{-\frac{p+\nu}{2}}} \\ &= \frac{\left(\frac{p+\nu}{2\nu} \right)}{\left[1 + \frac{t}{\nu} \right]}. \end{aligned}$$

Thus the proof follows readily. □

3.2 ARE of W_{np}

The Pitman asymptotic relative efficiency of two test statistics S and T, say ARE(S, T), is defined in the following fashion. Let $H_n : \boldsymbol{\theta} = \boldsymbol{\theta}_n$ be a sequence of alternatives approaching the null hypothesis, which, without loss of generality, can be taken as $H_o : \boldsymbol{\theta} = \mathbf{0}$, and consider the two test sequences $\{S_{m_n}\}$ and $\{T_{m_n^*}\}$ associated with this set of alternatives. Here, m_n and m_n^* are the sample sizes required for the two sequences of tests to satisfy

$$0 < \alpha < \lim_{n\to\infty} \beta_{S_{m_n}} = \lim_{n\to\infty} \beta_{T_{m_n^*}} < 1$$

where α is the the common limiting significance level of the two test sequences and $\beta_{S_{m_n}}$, $\beta_{T_{m_n^*}}$ are the powers of $S_{m_n}(\boldsymbol{\theta}_n)$ and $T_{m_n^*}(\boldsymbol{\theta}_n)$ (i.e. the probabilities that the tests S_{m_n} and $T_{m_n^*}$ reject H_o when $\boldsymbol{\theta} = \boldsymbol{\theta}_n$.). Then the asymptotic relative efficiency of S relative to T is

$$ARE(S,T) = \lim_{n\to\infty} \frac{m_n^*}{m_n}, \tag{3.10}$$

provided the limit exists and is the same for all $\{\boldsymbol{\theta}_n\}$, $\{m_n\}$ and $\{m_n^*\}$. Note that, if the test sequences $\{S_{m_n}\}$ and $\{T_{m_n^*}\}$ are asymptotically non-central chi-square with p degrees of freedom and non-centrality parameters, λ_S and λ_T, respectively, then

$$ARE(S,T) = \frac{\lambda_S}{\lambda_T}. \tag{3.11}$$

(see Hannan, 1956).

Before, the ARE's of W_{np} with respect to Hotelling's T^2 are obtained, the non-null asymptotic distributions of Hotelling's T^2 for the two families of distributions are given in the following, well-known, theorems (see Möttönen et al., 1998).

Theorem 3.2.1. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from the power family of distributions, with* $\boldsymbol{\Sigma} = \mathbf{I}$, *then under the alternatives defined in (3.1), Hotelling's* T^2 *has a non-central* *chi-square distribution with p degrees of freedom and non-centrality parameter*

$$\delta_{1_T} = \mathbf{c}'\mathbf{c}. \tag{3.12}$$

Theorem 3.2.2. *If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid from the multivariate t-distribution family, with $\mathbf{\Sigma} = \mathbf{I}$, then under the alternatives defined in (3.1), Hotelling's T^2 has a non-central chi-square distribution with p degrees of freedom and non-centrality parameter*

$$\delta_{2_T} = \left(\frac{\nu - 2}{\nu}\right)\mathbf{c}'\mathbf{c}, \tag{3.13}$$

where ν is the degrees of freedom of the t-distribution.

Now, the ARE's of W_{np} with respect to Hotelling's T^2 are given in the following theorems.

Theorem 3.2.3. *If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid from the power family of distributions, then, for testing $H_o : \boldsymbol{\theta} = \mathbf{0}$ against the alternative in (3.1),*

$$ARE(W_{np}, T^2) = \left(\left(\frac{2\nu}{p^{1/2}c_o^{\nu}}\right)^2 \frac{E^2_{H_o}\left[\phi(G(R))R^{2\nu-1}\right]}{E_{H_o}\left[\phi^2(G(R))\right]}\right), \tag{3.14}$$

where

$$E_{H_o}\left[\phi(G(R))R^{2\nu-1}\right] = \\ \frac{2K_p\pi^{p/2}}{\Gamma(p/2)}\left[\int_0^{\infty} \phi\left(\frac{2K_p\pi^{p/2}}{\Gamma(p/2)}\int_0^r t^{p-1}e^{-\left(\frac{t^2}{c_o}\right)^{\nu}}dt\right) r^{2\nu+p-2}e^{-\left(\frac{r^2}{c_o}\right)^{\nu}}dr\right], \tag{3.15}$$

and

$$E_{H_o}\left[\phi^2(G(R))\right] = \int_0^1 \phi^2(u)du. \tag{3.16}$$

Proof. Without loss of generality, take $\mathbf{\Sigma} = \mathbf{I}$. Result (3.14) follows immediately from corollary 3.1.1, theorem 3.2.1, and result (3.11). To prove (3.15) and (3.16), recall that $G(\cdot)$ represents that cumulative distribution function of the R_i's. Thus, for the power family of distributions, $t = R^2$ has the density

$$l(t) = \frac{K_p\pi^{p/2}}{\Gamma(p/2)}t^{\frac{p}{2}-1}\exp\left(-\frac{t^{\nu}}{\mathbf{c}_o^{\nu}}\right), \tag{3.17}$$

where K_p and $\mathbf{c}_o$ are defined in (3.3). Let $R = \sqrt{t}$ then $dR = \frac{1}{2t^{1/2}}dt$ and the density of R is

$$g(r) = \frac{2K_p\pi^{p/2}}{\Gamma(p/2)} r^{p-1} \exp\left(-\frac{r^{2v}}{\mathbf{c}_o^v}\right). \tag{3.18}$$

Thus, (3.15) follows readily from (3.18). To complete the proof, write

$$E_{H_o}\left[\phi^2(G(R))\right] = \int_0^\infty \phi^2(G(r))g(r)dr.$$

Let $u = G(r)$, then $du = g(r)dr$ and (3.16) is easily established. □

Theorem 3.2.4. *If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid from the t-distribution family of distributions,then, for testing $H_o : \boldsymbol{\theta} = \mathbf{0}$ against the alternative in (3.1),*

$$ARE(W_{np}, T^2) = \left(\frac{\nu}{\nu - 2}\right)\left(\left(\frac{p+\nu}{\sqrt{p}\nu}\right)^2 \frac{E^2_{H_o}\left[\phi(G(R))R/(1+\frac{1}{\nu}R^2)\right]}{E_{H_o}\left[\phi^2(G(R))\right]}\right), \tag{3.19}$$

where

$$E_{H_o}\left[\phi(G(R))R/(1+\frac{1}{\nu}R^2)\right] =$$
$$\frac{2\Gamma\left(\frac{p+\nu}{2}\right)}{\Gamma(p/2)\Gamma(\nu/2)\nu^{p/2}}\int_0^\infty \phi\left[\frac{2\Gamma\left(\frac{p+\nu}{2}\right)}{\Gamma(p/2)\Gamma(\nu/2)\nu^{p/2}}\int_0^r \frac{t^{p-1}}{\left[1+\frac{t^2}{\nu}\right]^{\frac{p+\nu}{2}}}dt\right]\frac{r^{p-1}}{\left[1+\frac{r^2}{\nu}\right]^{\frac{p+\nu+2}{2}}}dr \tag{3.20}$$

and $E_{H_o}\left[\phi^2(G(R))\right]$ is as defined in (3.16).

Proof. Without loss of generality, take $\boldsymbol{\Sigma} = \mathbf{I}$. Result (3.19) follows immediately from corollary 3.1.2, theorem 3.2.2, and result (3.11). To prove (3.20), let $t = \frac{r^2}{p}$, then t is distributed as an F-distribution with p numerator degrees of freedom and ν denominator degrees of freedom (see Möttönen et al., 1998, p.125). Thus, the density function of t is

$$l(t) = \frac{\Gamma\left(\frac{p+\nu}{2}\right)}{\Gamma(p/2)\Gamma(\nu/2)}(p/\nu)^2 \frac{t^{\frac{p-2}{2}}}{\left[1+(p/\nu)t\right]^{\frac{p+\nu}{2}}}.$$

Taking $r = \sqrt{pt}$, then $dr = \frac{p}{2}(pt)^{-1/2}dt$ and R has a density of the form:

$$g(r) = \left(\frac{2\Gamma\left(\frac{p+\nu}{2}\right)}{\Gamma(p/2)\Gamma(\nu/2)\nu^{p/2}} \right) \frac{r^{p-1}}{\left[1 + \frac{r^2}{\nu}\right]^{\frac{p+\nu}{2}}}. \tag{3.21}$$

Note that (3.20) follows readily from (3.21). This completes the proof. □

Table 3.1 displays the Pitman asymptotic relatives efficiencies of the new signed-rank statistic, W_{lnp}, Randles (2000) sign test, S_n, the signed-rank statistic due to Peters and Randles (1990), PR_n, and the signed sum statistic due to Jan and Randles (1994), JR_n, relative to Hotelling's T^2, for the case when the $\mathbf{Y}$'s come from the power family of distributions. Table 3.2 contains the efficiencies of W_{lnp}, S_n, PR_n, and the signed-rank test of Hettmansperger et al. (1994), HNO_n, relative to Hotelling's T^2, for the case when the $\mathbf{Y}$'s come from the family of multivariate t-distributions. Note also that the interdirection sign test of Randles (1989) has the same efficiencies as S_n and the test by Hössjer and Croux (1995) has the same efficiencies as PR_n.

It is clear from tables 3.1 and 3.2 that the new signed-rank test statistic has strong efficiencies among a variety of distributions, ranging from very heavy-tailed distribution to very light-tailed ones. It is also seen that these efficiencies do not deteriorate when the dimension increases as is the case for the test proposed by Peters and Randles (1990). For normal alternatives (Table 3.1 with $\nu = 1$ or Table 3.2 with degrees of freedom going to ∞), W_{lnp} not only performs virtually as well as Hotelling's T^2 but also has higher efficiencies than the other tests, S_n, PR_n, JR_n, and HNO_n when $p \geq 2$ (with the exception of PR_n, when $p = 2$). For the heavy-tailed distribution of the power family (that is $\nu < 1$), W_{lnp} has higher efficiencies than both PR_n and JR_n but slightly lower efficiencies than those of S_n, especially when the dimension increases. For the light-tailed distributions of this family (that is $\nu > 1$), the efficiencies of W_{lnp} exceed those of JR_n and S_n but are less than those of PR_n. However, this slight advantage of PR_n in the

Table 3.1: Asymptotic Relative Efficiencies Under the Power Family of Distributions

		ν				
Dimension	Test	.2	.5	1	2	5
1	W_{lnp}	11.949	1.500	.955	.873	.907
	PR_n	11.949	1.500	.955	.873	.907
	JR_n	11.949	1.500	.955	.873	.907
	S_n	127.451	2.000	.637	.411	.347
2	W_{lnp}	5.739	1.332	.971	.931	1.001
	PR_n	3.003	1.130	.985	1.051	1.221
	JR_n	4.188	1.296	.966	.883	.888
	S_n	11.597	1.500	.785	.590	.519
3	W_{lnp}	3.547	1.240	.978	.950	1.025
	PR_n	1.818	1.000	.975	1.099	1.346
	JR_n	2.741	1.211	.974	.898	.874
	S_n	5.346	1.333	.849	.688	.620
4	W_{lnp}	2.688	1.186	.982	.961	1.034
	PR_n	1.402	.940	.961	1.109	1.395
	JR_n	2.168	1.164	.978	.912	.878
	S_n	3.593	1.250	..884	.749	.687
5	W_{lnp}	2.247	1.151	.985	.968	1.038
	PR_n	1.197	.900	.949	1.106	1.412
	JR_n	1.902	1.136	.981	.922	.896
	S_n	2.816	1.200	.905	.790	.734
10	W_{lnp}	1.523	1.077	.993	.987	1.040
	PR_n	.875	.825	.907	1.061	1.365
	JR_n	1.388	1.071	.990	.953	.934
	S_n	1.702	1.100	.951	.885	.849
20	W_{lnp}	1.234	1.037	.997	.997	1.033
	PR_n	.760	.788	.869	.995	1.245
	JR_n	1.181	1.036	.994	.974	.962
	S_n	1.310	1.050	.975	.940	.920

Entries: Pitman asymptotic relative efficiencies with respect to Hotelling's T^2.
Distribution = Power family.
W_{lnp} = New procedure with the linear $\phi(\cdot)$ function.
PR_n = Peters and Randles procedure.
JR_n = Jan and Randles procedure.
S_n = Randles (2000) procedure.

Table 3.2: Asymptotic Relative Efficiencies Under Multivariate t-Distribution

		Degrees of freedom				
Dimension	test	3	6	10	20	∞
1	W_{lnp}	1.900	1.164	1.054	.997	.955
	PR_n	1.900	1.164	1.054	.997	.955
	HNO_n	1.900	1.164	1.054	.997	.955
	JR_n	1.900	1.164	1.054	.997	.955
	S_n	1.621	.879	.757	.690	.637
2	W_{lnp}	1.954	1.181	1.068	1.011	.971
	PR_n	1.748	1.123	1.041	1.005	.985
	HNO_n	2.026	1.196	1.064	.992	.937
	JR_n	1.952	1.185	1.069	1.011	.966
	S_n	2.000	1.084	.934	.851	.785
4	W_{lnp}	2.055	1.209	1.084	1.023	.984
	PR_n	1.533	1.018	.964	.952	.961
	HNO_n	2.173	1.241	1.088	1.004	.937
	JR_n	2.002	1.207	1.088	1.025	.978
	S_n	2.250	1.220	1.051	.958	.884
6	W_{lnp}	2.120	1.228	1.095	1.029	.989
	PR_n	1.423	.953	.911	.911	.938
	HNO_n	2.256	1.270	1.108	1.018	.947
	JR_n	2.024	1.218	1.095	1.031	.984
	S_n	2.344	1.271	1.094	.997	.920
10	W_{lnp}	2.194	1.251	1.107	1.037	.993
	PR_n	1.315	.882	.848	.857	.907
	HNO_n	2.346	1.304	1.132	1.037	.961
	JR_n	2.045	1.228	1.103	1.038	.990
	S_n	2.422	1.313	1.131	1.031	0.951

Entries: Pitman asymptotic relative efficiencies with respect to Hotelling's T^2.

Distribution = Multivariate t-distribution.
W_{lnp} = New procedure with the linear $\phi(\cdot)$ function.
PR_n = Peters and Randles procedure.
HNO_n = Hettmansperger, Nyblom and Oja (1994) procedure.
JR_n = Jan and Randles procedure.
S_n = Randles (2000) procedure.

light-tailed cases disappears slowly as the dimension increases. Note that as the dimension goes to infinity, W_{lnp} becomes S_n, and the efficiencies of W_{lnp}, JR_n, and S_n approach 1, while those of PR_n go to .75.

For the multivariate t-distribution family, W_{lnp} has higher efficiencies than PR_n for all degrees of freedom. This confirms the previous results since the t-distribution is heavier-tailed than the normal distribution. As compared to HNO_n, W_{lnp} has slightly smaller efficiencies when the degrees of freedom are less than 20 and larger efficiencies otherwise. The sign test, S_n, is designed for heavy-tailed cases and has slightly higher efficiencies than W_{lnp} as a result. The efficiencies of W_{lnp} are slightly larger than those of JR_n for small degrees of freedom and virtually the same for the higher ones. Here also, when the the dimension increases to infinity, the efficiencies of all the tests, with the exception of PR_n, approach one.

3.3 Monte Carlo Results

A Monte Carlo study for dimension $p = 2$, sample size $n = 30$, and significance level $\alpha = .05$ was conducted with the objective of investigating the performance of W_{lnp}. Also included in the study were Hotelling's T^2, the quadratic form, S_n, due to Randles (see 2.4), the bivariate signed rank test, BH, due to Brown and Hettmansperger. These four test statistics were compared under elliptical symmetric distributions with densities from the power family and the family of multivariate t-distributions. The distributions were located at $\boldsymbol{\theta} = (k\delta, k\delta), k = 0, 1, 2, 3$. The value δ was adjusted for different distributions to examine somewhat similar points on the power curves. All calculations and random variable generations were performed on a Unix workstation using OX, an object oriented language.

To generate observations from distribution in the power family, recall that ,when $\boldsymbol{\theta} = \mathbf{0}$ and $\boldsymbol{\Sigma} = \mathbf{I}$, R^2 has the distribution described in (3.17). Taking $W = (R^2/c_o)^\nu$, it can be easily seen that W has Gamma(p/2ν,1) distribution. Moreover, recall that each observation can be written as $\mathbf{Y} = R\mathbf{U}$, where R is independent of $\mathbf{U}$ which is uniformly distributed on the unit-sphere (see Johnson (1987)). Thus to generate the $\mathbf{Y}$'s, the uniform vector $\mathbf{U}$ is generated via: $\mathbf{U} = \mathbf{Z}/||\mathbf{Z}||$, where $\mathbf{Z}$ has the bivariate standard normal distribution, and R via $R = \sqrt{c_o} w^{1/2\nu}$, where c_o is defined in (3.3). The entries in the tables represent the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution. In Table 3.3, the results from the first Monte Carlo study are presented. As indicated, 3000 samples of size 30 are obtained from five members of the power family of distributions. Heavy-tailed distributions, $\nu = .10$ and $\nu = .20$, the bivariate normal distribution, $\nu = 1$, and light-tailed distributions, $\nu = 2.0$ and $\nu = 10.0$, are included. Table 3.4 contains the results from the second Monte Carlo study where the underlying distribution is chosen from the bivariate family of t-distributions with degrees of freedom of 1, 3, 10 and 20.

The results of the Monte Carlo studies agree with those of the previous section. For the bivariate normal alternatives, Hotelling's T^2, W_{lnp} and BH have very similar powers, with the power of Hotelling's T^2 being slightly larger than the others. For the heavy-tailed distributions, S_n has a slightly larger power than that of W_{lnp} and BH, with Hotelling's T^2' s power significantly lower (see Table 3.3, case of $\nu = .10$). For the light-tailed distribution, W_{lnp} and Hotelling's T^2 maintained a good performance while that of BH and S_n declined. In summary, it can be concluded that W_{lnp} maintained a solid performance under the heavy-tailed, light-tailed and multivariate normal distributions, while the other tests' performances declined in one of the three cases. For example, Hotelling's T^2 didn't perform well under very heavy-tailed distributions, the performance of quadratic form, S_n, was not good for light-tailed cases, and the powers of BH_n declined for distribution with lighter tails than the normal case.

In the second Monte Carlo study, the dimension was increased to $p = 4$, the number of iterations to 5000, and the sample size to $n = 50$. In this study, the generalization of BH to the multivariate case; that is the test due to Hettmansperger, Nyblom, and Oja; was not included because it is very difficult to calculate the test statistic for dimensions larger than two. Distributions from the power family and the family of multivariate t-distributions are considered. The results are presented in Tables 3.5 and 3.6. For the normal case, the three tests perform very well with a slight advantage to T^2 and W_{lnp} over S_n. For heavy-tailed distributions from the power family, the performance of Hotelling's T^2 deteriorates dramatically, while the performance of both W_{np} and S_n stays good, with S_n having the higher power. For the light-tailed cases from the power family, W_{lnp} and T^2 have similar powers that are higher than those of S_n. For the extreme heavy-tailed cases from the family of t-distributions, it is clear that S_n and W_{lnp} have a much better performance than T^2, and as the degrees of freedom increases, the powers of the three tests become virtually the same. Once again, it is demonstrated that W_{lnp} maintains a very good performance for all distributions considered, while the other tests show a decline in their performances in some cases.

While the conclusions from the Monte Carlo study in dimension 4 are similar to those from the bivariate case, it is worth noting that the time it took for each run of the first Monte Carlo study, that is 3000 iterations of bivariate samples of size 30, is more than 36 hours where, on the other hand, a run from the second Monte Carlo study, that is 5000 iterations of samples of size 50 from quadrivariate distributions took less than 5 minutes. The reason of this drop in computation time is the absence of the test statistic of Hettmasnperger et. al from the second Monte Carlo study. This difference in the computation time also demonstrates the ease with which one can calculate the new test statistic for data in any practical dimension.

In conclusion, the test comparisons, whether done by using the Pitman asymptotic relative efficiencies or Monte Carlo studies, demonstrate that the new proposed multivariate affine-invariant signed-rank test performs as well as, if not better than, other leading competitors. Moreover, this performance is robust to the types of distributions considered, which are taken from the power family and from the family of multivariate t-distributions. Most important is the fact that the new procedure is much easier to implement than tests found in the literature, with the exception of S_n, which, by the way, is a special case of W_{np}.

Table 3.3: Monte Carlo Results for the Bivariate Power Family of Distributions

Test Statistic	ν	Amount of Shift			
	.10	0	1Δ	2Δ	3Δ
T^2		0.0267	0.0627	0.1947	0.3697
W_{lnp}		0.0550	0.3617	0.7207	0.8937
BH		0.0560	0.3017	0.6557	0.8530
S_n		0.0527	0.5617	0.8890	0.9743
	.20				
T^2		0.0330	0.1033	0.3123	0.5643
W_{lnp}		0.0460	0.2213	0.5850	0.8433
BH		0.0433	0.2080	0.5753	0.8423
S_n		0.0413	0.3120	0.7267	0.9243
	1.0				
T^2		0.0480	0.1500	0.5100	0.8773
W_{lnp}		0.0463	0.1367	0.4770	0.8593
BH		0.0463	0.1350	0.4693	0.8487
S_n		0.0470	0.1287	0.4070	0.7727
	2.0				
T^2		0.0537	0.1410	0.4607	0.8430
W_{lnp}		0.0463	0.1247	0.4127	0.7940
BH		0.0487	0.1170	0.3843	0.7560
S_n		0.0500	0.1027	0.2993	0.6207
	10.0				
T^2		0.0493	0.1580	0.4937	0.8783
W_{lnp}		0.0437	0.1407	0.4560	0.8343
BH		0.0450	0.1203	0.3910	0.7660
S_n		0.0477	0.0987	0.2877	0.6087

Entries: the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Power family.
α = .05
$\mathbf{\Sigma}$ = $\mathbf{I}$.
Sample size = 30.
Dimension = 2.
T^2 = Hotelling's procedure
W_{lnp} = New procedure with the linear $\phi(\cdot)$ function.
BH = Brown and Hettmansperger (1989) procedure.
S_n = Randles (2000) procedure.

Table 3.4: Monte Carlo Results for the Bivariate t-Distributions

Test Statistic	df	Amount of Shift			
	1	0	1Δ	2Δ	3Δ
T^2		0.0130	0.0393	0.1117	0.2317
W_{lnp}		0.0460	0.1487	0.4383	0.7177
BH		0.0453	0.1420	0.4260	0.6980
S_n		0.0500	0.1757	0.5697	0.8443
	3				
T^2		0.0427	0.1440	0.4580	0.7797
W_{lnp}		0.0477	0.1713	0.5600	0.8810
BH		0.0503	0.1753	0.5737	0.8917
S_n		0.0477	0.1757	0.5697	0.8897
	10				
T^2		0.0420	0.1453	0.4903	0.8543
W_{lnp}		0.0387	0.1417	0.4887	0.8583
BH		0.0360	0.1463	0.4873	0.8567
S_n		0.0413	0.1280	0.4460	0.8103
	20				
T^2		0.0513	0.1433	0.4910	0.8510
W_{lnp}		0.0527	0.1363	0.4763	0.8457
BH		0.0513	0.1420	0.4730	0.8443
S_n		0.0527	0.1260	0.4207	0.7737

Entries: the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Multivariate t-distribution.
α = .05
$\mathbf{\Sigma}$ = $\mathbf{I}$.
Sample size = 30.
Dimension = 2.
T^2 = Hotelling's procedure.
W_{lnp} = New procedure with the linear $\phi(\cdot)$ function.
BH = Brown and Hettmansperger (1989) procedure.
S_n = Randles (2000) procedure.

Table 3.5: Monte Carlo Results for the Multivariate Power Family of Distributions

Test Statistic	ν	Amount of Shift			
	.10	0	1Δ	2Δ	3Δ
T^2		0.0280	0.0450	0.1194	0.2504
W_{lnp}		0.0436	0.1646	0.4868	0.7702
S_n		0.0426	0.2214	0.6116	0.8796
	.20				
T^2		0.0380	0.0922	0.3058	0.6142
W_{lnp}		0.0422	0.1658	0.5370	0.8688
S_n		0.0402	0.1988	0.6350	0.9266
	1.0				
T^2		0.0524	0.1486	0.5198	0.8992
W_{lnp}		0.0472	0.1298	0.4862	0.8800
S_n		0.0466	0.1254	0.4466	0.8432
	2.0				
T^2		0.0558	0.1486	0.5128	0.8904
W_{lnp}		0.0470	0.1270	0.4678	0.8570
S_n		0.0516	0.1132	0.3908	0.7678
	5.0				
T^2		0.0510	0.1436	0.5272	0.9050
W_{lnp}		0.0428	0.1258	0.4870	0.8836
S_n		0.0472	0.1046	0.3704	0.7664

Entries: the proportion of times out of 5000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Power family.
α = .05
$\mathbf{\Sigma}$ = $\mathbf{I}$.
Sample size = 50.
Dimension = 4.
T^2 = Hotelling's procedure
W_{lnp} = New procedure with the linear $\phi(\cdot)$ function.
S_n = Randles (2000) procedure.

Table 3.6: Monte Carlo Results for the Multivariate t-Distributions

Test Statistic	df	Amount of Shift			
	1	0	1Δ	2Δ	3Δ
T^2		0.0136	0.0262	0.0772	0.1748
W_{lnp}		0.0430	0.1218	0.4382	0.7630
S_n		0.0432	0.1508	0.5442	0.8698
	3				
T^2		0.0370	0.0946	0.3358	0.6752
W_{lnp}		0.0452	0.1186	0.4514	0.8164
S_n		0.0448	0.1258	0.4866	0.8498
	10				
T^2		0.0488	0.1300	0.4640	0.8346
W_{lnp}		0.0454	0.1216	0.4636	0.8484
S_n		0.0444	0.1172	0.4520	0.8372
	20				
T^2		0.0526	0.1518	0.5310	0.8978
W_{lnp}		0.0446	0.1428	0.5122	0.8872
S_n		0.0402	0.1342	0.4944	0.8688

Entries: the proportion of times out of 5000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Multivariate t-distribution.
α = .05
$\mathbf{\Sigma}$ = $\mathbf{I}$.
Sample size = 50.
Dimension = 4.
T^2 = Hotelling's procedure
W_{lnp} = New procedure with the linear $\phi(\cdot)$ function.
S_n = Randles (2000) procedure.

CHAPTER 4
SIMPLE REPEATED-MEASURES DESIGNS

In the remainder of this dissertation, the new proposed signed-rank test is applied to the single-factor repeated-measures designs. A literature review following the introduction of such designs is presented in this chapter.

To introduce single-factor repeated-measures designs, let $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ be iid as $\mathbf{Y}$, where $\mathbf{Y} = (Y_1, \cdots, Y_p)'$ is from a p-dimensional, $p \geq 2$, absolutely continuous population. Each observation, $\mathbf{Y}_i$, is regarded as repeated measures, one observation for each of the p treatments, taken on the i^{th} experimental unit (subject). The treatments are assumed to be fixed while the subjects are taken as random. Moreover, a subject by treatment interaction is included to the model. Thus, the general mixed model with a subject by treatment interaction is used. In vector form, the model is

$$\mathbf{Y}_i = \beta_i \mathbf{1}_p + \boldsymbol{\tau} + (\boldsymbol{\beta\tau})_i + \boldsymbol{\epsilon}_i, \qquad i = 1, \cdots, n \tag{4.1}$$

where $\mathbf{1}_p$ is the $p \times 1$ vector of 1's, $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_p)'$ is the vector of fixed treatment effects, β_i represents the random effect of the ith subject, $\boldsymbol{\beta\tau}_i$ denotes the vector of the ith subject by treatment interactions, and $\boldsymbol{\epsilon}_i$ is the vector of random error of the ith subject. The following assumptions are made. The β_i's are iid with mean 0 and variance σ_β^2, $\boldsymbol{\beta\tau}_i$'s are iid with mean $\mathbf{0}$ and a general variance-covariance matrix, and the $\boldsymbol{\epsilon}_i$'s are iid with mean $\mathbf{0}$ and variance-covariance matrix $\sigma_\epsilon^2 \mathbf{I}_p$, where $\mathbf{I}_p$ is the $p \times p$ identity matrix. Moreover, the random variables β_i, $\boldsymbol{\beta\tau}_i$, $\boldsymbol{\epsilon}_i$ for $i = 1, \ldots, n$ are all mutually independent. The test of equal treatment effects is of interest. This test can be stated as

$$H_o : \tau_1 = \cdots = \tau_p \quad \text{versus} \quad H_a : \tau_i \neq \tau_j \text{ for some } i \neq j. \tag{4.2}$$

Several procedures can be considered for such designs. Starting with parametric procedures, two classical approaches are often recommended. The first procedure, due to Hsu (1938), is an application of Hotelling's T^2. Hsu argues that the test in (4.2) can be carried out by applying Hotelling's T^2 (described in Chapter 1) to the $(p-1)$-variate random vectors, $\mathbf{X}_i$'s, obtained from the $\mathbf{Y}_i$'s via

$$\mathbf{X}_i = (X_1, \cdots, X_{p-1})' = (Y_1 - Y_p, \cdots, Y_{p-1} - Y_p)' \qquad i = 1, \cdots, n.$$

Recall that Hotelling's procedure is affine-invariant, and if the $\mathbf{Y}$'s come from the p-variate normal distribution, T^2 will have a null distribution that is a multiple of the F-distribution with $p-1$ and $n-(p-1)$ numerator and denominator degrees of freedom, respectively. The affine-invariance property is important because it ensures that the performance of the test is not affected by the shape of the population variance-covariance matrix, as is the case with some of the tests considered later. The second procedure is the classical analysis of variance (ANOVA) F test. This procedure uses the $\mathbf{Y}$'s and the test statistic

$$F = \frac{n \sum_{j=1}^{p} (\bar{Y}_{\cdot j} - \bar{Y}_{\cdot\cdot})^2}{\frac{1}{n-1} \sum_{i=1}^{n} \sum_{j=1}^{p} (Y_{ij} - \bar{Y}_{i\cdot} - \bar{Y}_{\cdot j} + \bar{Y}_{\cdot\cdot})^2}, \tag{4.3}$$

where Y_{ij} denotes the jth element of the vector $\mathbf{Y}_i$, $\bar{Y}_{\cdot j} = (1/n) \sum_{i=1}^{n} Y_{ij}$, $\bar{Y}_{i\cdot} = (1/p) \sum_{j=1}^{p} Y_{ij}$, and $\bar{Y}_{\cdot\cdot} = (1/np) \sum_{i=1}^{n} \sum_{j=1}^{p} Y_{ij}$. When the $\mathbf{Y}_i$'s are (multivariate)-normally distributed, the null distribution of the ANOVA test is an exact F-distribution with $p-1$ and $(n-1)(p-1)$ degrees of freedom if and only if the population variance-covariance matrix of $\mathbf{Y}$, say $\mathbf{\Sigma} = (\sigma_{tt'})$, satisfies the Huynh and Feldt condition (see Huynh and Feldt, 1970). Huynh and Feldt give three forms to express this necessary and sufficient condition:

1. The elements $\sigma_{tt'}$ of the variance-covariance matrix $\mathbf{\Sigma}$ are of the form:

$$\sigma_{tt'} = \alpha_t + \alpha_{t'} + \lambda\delta_{tt'}, \tag{4.4}$$

   where $\lambda > 0$ and $\delta_{tt'} = 1$ if $t = t'$ and 0 otherwise.
2. All possible differences $Y_{ij} - Y_{ij'}$ $j \neq j'$ are equally variable.
3. $\varepsilon = 1$, where ε is a function of the elements of $\mathbf{\Sigma}$ defined by

$$\varepsilon = \frac{p^2(\bar{\sigma}_d - \bar{\sigma}_{..})^2}{(p-1)\left(\sum_{t=1}^{p}\sum_{t'=1}^{p}\sigma_{tt'}^2 - 2p\sum_{t=1}^{p}\bar{\sigma}_{t\cdot}^2 + p^2\bar{\sigma}_{..}^2\right)},$$

   where $\bar{\sigma}_d = (1/p)\sum_{t=1}^{p}\sigma_{tt}$, $\bar{\sigma}_{t\cdot} = (1/p)\sum_{t'=1}^{p}\sigma_{tt'}$, and $\bar{\sigma}_{..} = (1/p)\sum_{t=1}^{p}\bar{\sigma}_{t\cdot}$

Note that ε is the multiplicative adjustment factor for the degrees of freedom proposed by Greenhouse and Geisser (1959). Any matrix whose elements satisfy (4.4) is referred to as a matrix of Type H. Two disadvantages of using the ANOVA F test is the dependence of its performance on the structure of the variance-covariance matrix, and, unlike Hotelling's T^2, its lack of the affine-invariance property. However, if the underlying distribution is p-variate normal with H-type variance covariance matrix, the F test will have a greater power than the Hotelling-Hsu T^2 test for the same alternatives (see Morrison, 1972).

Morrison (1972) developed several statistics to test (4.2) under various assumptions on the variance-covariance matrix $\mathbf{\Sigma}$. The statistics included the usual Hotelling-Hsu statistic and the classical F test as special cases. Morrison also measured the efficacies of the tests in terms of the average squared lengths of their simultaneous confidence intervals. He demonstrated that his tests have higher efficacies than that of Hotelling-Hsu T^2. These differences in efficacies decrease when the sample size increases, and do increase as the dimension increases. Yet, Morrison only considered the case where the observations come from a multivariate normal distribution.

Nonparametric competitors to Hotelling's T^2 appeared as early as 1937 and 1939 when Friedman developed his well known *method of ranks* procedure and Kendall and Babington proposed an equivalent test, respectively. Assuming no ties between observations within a subject, Friedman's test statistic has the form

$$T = \frac{12}{np(p+1)} \sum_{j=1}^{p} \left(\sum_{i=1}^{n} R_{ij} \right)^2 - 3n(p+1),$$

where R_{ij} denotes the rank of Y_{ij} among $Y_{i1}, \cdots, Y_{ip}$. If the components of each observation vector are exchangeable, then T is distribution-free. Under this H_o, T has an asymptotic chi-square distribution with $(p-1)$ degrees of freedom. An alternative approximation of Friedman's test uses the statistic

$$F_R = \frac{(n-1)T}{n(p-1) - T},$$

which is then compared with the usual F distribution with $(p-1)$ and $(n-1)(p-1)$ numerator and denominator degrees of freedom, respectively. Jensen (1977) gave error bounds on the large sample approximation of F_R. Iman and Davenport (1980) showed that the chi-square large sample approximation can be inaccurate and recommended using the F approximation. In their paper, Iman and Davenport also proposed two new approximations of Friedman's test. The first approximation is a linear combination of the T and F_R, while the second approximations is obtained by using the ranks, R_{ij}, to estimate the degrees of freedom for F_R. It is important to note that Friedman's procedure only uses within-subject information and ignores between-subjects information.

In order to reduce the effect of subject-to-subject variation, Koch (1969) proposed using a test statistic based on the ranks of the aligned observations. These observations are obtained by subtracting the subject average from each component of that subject's response vector. After this centering, the entire set of the np observations is ranked. Koch aligned-ranks test statistic has the form

$$KAR = \frac{n\sum_{j=1}^{p}(\bar{R}^a_{\cdot j} - \frac{np+1}{2})^2}{\frac{1}{n(p-1)}\sum_{i=1}^{n}\sum_{j=1}^{p}(R^a_{ij} - \bar{R}^a_{i\cdot})^2},$$

where R^a_{ij} = rank of $(Y_{ij} - \bar{Y}_{i\cdot})$ among $Y_{11} - \bar{Y}_{1\cdot}, \cdots, Y_{np} - \bar{Y}_{n\cdot}$, $\bar{R}^a_{\cdot j} = \frac{1}{n}\sum_{i=1}^{n} R^a_{ij}$, and $\bar{R}^a_{i\cdot} = \frac{1}{p}\sum_{j=1}^{p} R^a_{ij}$. If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid, then under the null hypothesis of the exchangeability of the components of $\mathbf{Y}_i$, the statistic KAR has an asymptotic chi-square distribution with (p-1) degrees of freedom.

In an attempt to retrieve some of the between-subjects information, Quade (1979) presented a class of procedures of weighted within-subject rankings. The class contains, as a special case, a p-sample extension of Wilcoxon (1945) signed-rank test. The idea was to give larger weights to subjects that show the largest variability among the treatment responses for that subject. Thus, if R_{ij} is the rank of Y_{ij} among $Y_{i1}, \cdots, Y_{ip}$, $D_i = D(\mathbf{Y}_i)$, a location-free statistic that measures the variability within the components of Y_i, and Q_i the rank of D_i among $D_1, \cdots, D_n$, then Quade's procedure has the form:

$$W_Q = \frac{72S}{np(n+1)(p+1)(2n+1)} - \frac{9n(n+1)(p+1)}{2(2n+1)}, \tag{4.5}$$

where

$$S = \sum_{j=1}^{p}\left(\sum_{i=1}^{n} Q_i R_{ij}\right)^2.$$

Under the null hypothesis of the exchangeability of the components of the $\mathbf{Y}_i$, W_Q is strictly distribution-free, and has an asymptotic chi-square distribution with $(p-1)$ degrees of freedom. In a Monte Carlo study, Silva (1977) not only found improved efficiencies of the weighted-rankings over unweighted but also demonstrated that the choice of the measure D had a trivial effect on the performance of W_Q.

In an attempt to find natural nonparametric analogs for Hotelling's T^2 and the ANOVA F test, Iman, Hora and Conover (1984) and Agresti and Pendergast (1986) used the rank transformation approach. This approach consists of ranking all observations from 1 to np and replacing the data with their corresponding ranks in calculating the parametric tests.

Iman, Hora and Conover (1984) proposed computing the ANOVA F test using these ranks. That is the test statistic is defined as

$$F_{RT} = \frac{n\sum_{j=1}^{p}(\bar{R}^*_{.j} - \bar{R}^*_{..})^2}{\frac{1}{n-1}\sum_{i=1}^{n}\sum_{j=1}^{p}(R^*_{ij} - \bar{R}^*_{i.} - \bar{R}^*_{.j} + \bar{R}^*_{..})^2}, \tag{4.6}$$

where R^*_{ij} denotes the rank of Y_{ij} among $Y_{11}, \cdots, Y_{np}$, $\bar{R}^*_{.j} = (1/n)\sum_{i=1}^{n} R^*_{ij}$, $\bar{R}^*_{i.} = (1/p)\sum_{j=1}^{p} R^*_{ij}$, and $\bar{R}^*_{..} = (1/np)\sum_{i=1}^{n}\sum_{j=1}^{p} R^*_{ij}$. Iman, Hora and Conover considered a null hypothesis of the form

$$H_o : F_{ij} = F_i \text{ for } i = 1, \cdots, n \text{ and } j = 1, \cdots, p,$$

where F_{ij} is the continuous distribution function of the random variable Y_{ij}. Under the assumption that the Y_{ij}'s are mutually independent and for some regularity conditions on F_i, Iman, Hora, and Conover showed that the null asymptotic distribution of $(p-1)F_{RT}$ is that of a chi-square distribution with $(p-1)$ degrees of freedom. Note also that conditioned on the ranks given to each subject, F_{RT} is distribution-free. Iman, Hora, and Conover (1984), and later Hora, and Iman (1988) compared the rank transformation test to the ANOVA F test, Friedman's test, Quade's weighted-ranking test, and the aligned-rank test, through Monte Carlo Studies and ARE calculations. They concluded that, for data from the normal distribution, all the test procedures discussed had higher powers compared to Friedman's test, but these differences in power decreased as the dimension increased. Data generated from the lognormal distributions showed the superiority of nonparamteric tests, with the rank transformed test having the highest power, over the ANOVA F test. This finding is due to the fact that lognomal distributions produce outliers. For light-tailed cases, the ANOVA F test and Quade's test had the highest powers, while, for heavy-tailed cases, Friedman's test and the rank transformation test were the best to use.

Agresti and Pendergast (1986) suggested using this single overall ranking of the observations and then computing the Hotelling-Hsu statistic using these ranks. They considered the case where the null hypothesis of no treatment effects is expressed as $F_1 \equiv F_2 \equiv \cdots \equiv F_p$, where $F_1, \cdots, F_p$ are the one dimensional marginals distributions of $\mathbf{Y} = (Y_1, \cdots, Y_p)$. If R^*_{ij} is defined as the rank of Y_{ij} among $Y_{11}, \cdots, Y_{np}$, then the rank transformation version of the Hotelling-Hsu statistic is of the form

$$T^2_{RT} = n(\mathbf{B}\bar{\mathbf{R}}^*)'(\mathbf{BSB}')^{-1}(\mathbf{B}\bar{\mathbf{R}}^*),$$

where

$$\mathbf{B} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix},$$

$\bar{\mathbf{R}}^* = n^{-1}\sum_{i=1}^n \mathbf{R}^*_i$, $\mathbf{R}^*_i = (R^*_{i1}, \cdots, R^*_{ip})'$, and $\mathbf{S} = (s_{ab})$, where $s_{ab} = \sum_{i=1}^n (R^*_{ia} - \bar{R}^*_{\cdot a})(R^*_{ib} - \bar{R}^*_{\cdot b})/(n-p+1)$. If the vector $\bar{\mathbf{R}}^*$ is normally distributed, the null asymptotic distribution T^2_{RT} is chi-square with $(p-1)$ degrees of freedom. Agresti and Pendergast showed, through Monte Carlo results, that it was reasonable to approximate the null distribution of $T^2_{RT}/(p-1)$ as an F distribution with $(p-1)$ numerator degrees of freedom and $n-(p-1)$ denominator degrees of freedom.

Agresti and Pendergast (1986) proposed another statistic that made use of this joint ranking. The null hypothesis was considered as the exchangeability of the components of the $\mathbf{Y}_i$. Their statistic included Koch's statistic and the ANOVA F

rank transformation statistic as special cases and is based on

$$AP = \frac{n \sum_{j=1}^{p} (\bar{R}^*_{\cdot j} - \frac{np+1}{2})^2}{\sigma^2(1-\rho)}$$

where $\bar{R}^*_{\cdot j} = n^{-1} \sum_{i=1}^{n} R^*_{ij}$, $\rho = \text{Corr}(R^*_{ij}, R^*_{ij'})$, $j \neq j'$, and $\sigma^2 = \text{Var}(R^*_{ij})$. They argued that if the vector of the the first (p-1) treatment averages is normally distributed, the null asymptotic distribution of AP is chi-square with $(p-1)$ degrees of freedom. Kepner and Robinson (1988) not only gave sufficient conditions under which the vector $(\bar{R}^*_{\cdot 1}, \cdots, \bar{R}^*_{\cdot p-1})$ is normally distributed but also suggested a suitable estimate of $\sigma^2(1-\rho)$ which can be used in order to find conditional but exact p-values of the test for small sample sizes. Moreover, Ernst and Kepner (1993) investigated, through a Monte Carlo study, the performance of Koch's test, the rank transformation statistic (4.6), Friedman's test and the ANOVA F test. They observed that the Koch's statistic was better compared to its chi-square limiting distribution as opposed to its F approximating limiting distribution and the converse holds for the rank transformation ANOVA F test.

Jan and Randles (1996) proposed applying the one-sample interdirection sign test of Randles (1989) and the one-sample interdirection signed-rank test of Peters and Randles (1990) to this repeated measures design problem. They formed the (p-1) vectors $\mathbf{X}_i = (X_{i1}, \cdots, X_{ip-1})' = (Y_{i1} - Y_{ip}, \cdots, Y_{i\,p-1} - Y_{ip})'$ and then computed the interdirection sign and signed-rank statistics on the transformed observations. The test based on the sign test has the form

$$RIS = \frac{p-1}{n} \sum_{i=1}^{n} \sum_{j=1}^{p} \cos(\pi \hat{p}_{ik}),$$

where

$$\begin{aligned} \hat{p}_{ik} &= \frac{C_{ik} + d_n}{\binom{n}{p-2}} \text{ if } i \neq k, \\ &= 0 \text{ if } i = k, \end{aligned}$$

$dn = \frac{1}{2}[\binom{n}{p-2} - \binom{n-2}{p-2}]$, and C_{ik} denotes the interdirection of the vectors $\mathbf{X}_i$, $\mathbf{X}_k$. The form of the second test, which will be denoted by JR_n, can be easily found from (1.3). If the $\mathbf{Y}_i$'s are iid from a p-variate elliptically symmetric distribution then the null limiting distribution of RIS and also of JR_n is chi-square with $(p-1)$ degrees of freedom.

In the next chapter, the sign test of Randles (2000) and the new signed-rank test defined in Chapter 2 are applied to the simple repeated-measure setting. The null asymptotic distributions of the sign test and the signed-rank test are investigated under the assumption that the observations come from a directionally symmetric distribution and a symmetric distribution, respectively. In Chapter 6, the asymptotic distributions of these tests are obtained for a sequence of alternatives approaching the null. These limiting distributions are then used to find the Pitman asymptotic relative efficiencies of the tests. Comparisons between these tests and Jan and Randles (1996) signed-rank test is done based on the ARE findings. Also, in Chapter 6, comprehensive Monte Carlo studies are presented. These Monte Carlo studies include most of the tests described in the literature review and are aimed on comparing all these procedures and the new procedures discussed in Chapter 5. Finally, general comments and recommendations based on this dissertation are given.

CHAPTER 5
MULTIVARIATE SIGN AND SIGNED-RANK TESTS APPLIED TO THE SIMPLE REPEATED-MEASURES DESIGNS

5.1 Introduction

Simple repeated-measures designs have experimental settings in which n subjects are measured on p occasions with no replication. Moreover, the subjects are treated as random, while the repeated factor is considered fixed. Letting $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$, be iid as $\mathbf{Y}$, where $\mathbf{Y} = (Y_1, \cdots, Y_p)'$ is from a p-dimensional, $p \geq 2$, absolutely continuous population, this mixed model can be formulated as

$$\mathbf{Y}_i = \beta_i \mathbf{1}_p + \boldsymbol{\tau} + (\boldsymbol{\beta\tau})_i + \boldsymbol{\epsilon}_i, \qquad i = 1, \cdots, n \tag{5.1}$$

where $\mathbf{1}_p$ is the $p \times 1$ vector of 1's, $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_p)'$ is the vector of fixed treatment effects, β_i represents the random effect of the ith subject, $\boldsymbol{\beta\tau}_i$ denotes the vector of the ith subject by treatment interactions, and $\boldsymbol{\epsilon}_i$ is the vector of random error of the ith subject. It is assumed that the β_i's are iid with mean 0 and variance σ_β^2, $\boldsymbol{\beta\tau}_i$'s are iid with mean $\mathbf{0}$ and a general variance-covariance matrix, and the $\boldsymbol{\epsilon}_i$'s are iid with mean $\mathbf{0}$ and variance-covariance matrix $\sigma_\epsilon^2 \mathbf{I}_p$, where $\mathbf{I}_p$ is the $p \times p$ identity matrix. Moreover, the random variables β_i, $\boldsymbol{\beta\tau}_i$, $\boldsymbol{\epsilon}_i$ for $i = 1, \ldots, n$ are all mutually independent. The test of equal treatment effects,

$$H_o : \tau_1 = \cdots = \tau_p \text{versus} \quad H_a : \tau_i \neq \tau_j \text{ for some } i \neq j, \tag{5.2}$$

is of interest.

To apply the sign test of Randles (2000) and the new signed-rank test to this repeated-measures setting, the connection between this problem and the one-sample location problem is first established.

Consider the following transformation:

$$\mathbf{X}_i = (X_{i1}, \cdots, X_{i\,p-1})' = (Y_{i1} - Y_{ip}, \cdots, Y_{i\,p-1} - Y_{ip})' = \mathbf{C}\mathbf{Y}_i, \; i = 1, \cdots, n, \tag{5.3}$$

where $\mathbf{C}$ is the $(p-1) \times p$ matrix,

$$\mathbf{C} \;=\; \begin{bmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & \ddots & & \vdots \\ & & & 1 & -1 \end{bmatrix}. \tag{5.4}$$

Under this transformation, the location parameter of the $\mathbf{X}$'s is

$$\boldsymbol{\theta} = \mathbf{C}\boldsymbol{\tau} = [\tau_1 - \tau_p, \cdots, \tau_{p-1} - \tau_p].$$

Hence, using the p-variate $\mathbf{Y}$'s to test (5.2) is transformed to using the $(p-1)$-variate $\mathbf{X}$'s to test

$$H_o : \boldsymbol{\theta} = \mathbf{0} \text{ versus } H_a : \boldsymbol{\theta} \neq \mathbf{0},$$

which is nothing but a one-sample location problem.

5.2 The Test Statistics

In order to compute the quadratic form, S_n, due to Randles (2000), and the new signed-rank statistic, $W_{n\,p-1}$, on the transformed vectors, let $\hat{\mathbf{A}}_{\mathbf{X}}$ be the $(p-1) \times (p-1)$ nonsingular matrix that is computed using the $\mathbf{X}$'s and that satisfies

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i}{||\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i||}\right)\left(\frac{\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i}{||\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i||}\right)' = \frac{1}{p-1}\mathbf{I}_{p-1}.$$

Moreover, let Q_i be the rank of $\|\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i\|$ among $\|\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_1\|, \cdots, \|\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_n\|$ and define

$$\mathbf{V}_i = \frac{\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i}{\|\hat{\mathbf{A}}_{\mathbf{X}}\mathbf{X}_i\|}, \qquad \mathbf{V}_{si} = \phi\left(\frac{Q_i}{n+1}\right)\mathbf{V}_i,$$

where $\phi(\cdot)$ is a nonnegative, nondecreasing, uniformly bounded continuous function that may depend on the dimension $p-1$. Note that, for efficiency calculations and Monte Carlo studies, the $\phi(\cdot)$ function defined in (2.6) will be used. The quadratic forms are then

$$S_n = n(p-1)\bar{\mathbf{V}}'\bar{\mathbf{V}}, \tag{5.5}$$

and

$$W_{n\ p-1} = n\bar{\mathbf{V}}_s'(n^{-1}\sum_{i=1}^{n}\mathbf{V}_{si}\mathbf{V}_{si}')^{-1}\bar{\mathbf{V}}_s. \tag{5.6}$$

where

$$\bar{\mathbf{V}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{V}_i, \qquad \bar{\mathbf{V}}_s = \frac{1}{n}\sum_{i=1}^{n}\mathbf{V}_{si},$$

and H_o is rejected for large values. The test statistic S_n is distribution-free, under H_o for the class of distribution with elliptical directions and conditionally distribution-free for the broader class of directionally symmetric distributions. The quadratic form $W_{n\ p-1}$ is conditionally distribution-free for the class of symmetric distributions. Moreover, as mentioned earlier, both test statistics are affine-invariant (with respect to the $\mathbf{X}'s$) and thus their performances do not depend on the structure of the underlying population variance-covariance matrix as in the case of some of the tests discussed in chapter 4. Note that the particular contrast scheme in (5.3) is used without loss of generality. In fact, if $\mathbf{D}$ is any $(p-1)\times(p-1)$ nonsingular matrix, then using the transformation $\mathbf{DX}$ will produce exactly the same value for the test statistic, since S_n and $W_{n\ p-1}$ are affine-invariant. For example, many authors use the scheme:

$$\mathbf{X}_i^* = (X_{i1}^*, \cdots, X_{i\ p-1}^*)' = (Y_{i1} - Y_{i2}, Y_{i2} - Y_{i3}, \cdots, Y_{i\ p-1} - Y_{ip})' = \mathbf{C}^*\mathbf{Y}. \tag{5.7}$$

The matrix $\mathbf{C}^*$ in (5.7) can be written as $\mathbf{DC}$, where $\mathbf{D}$ is the $(p-1)\times(p-1)$ nonsingular matrix

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix},$$

thus, using the contrast scheme in (5.3) or the one in (5.7) will result in the same value of the test statistic.

5.3 Asymptotic Null Distribution

The following two lemmas are important in the development of the null limiting distributions of the test statistics.

Lemma 5.3.1. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from a symmetric distribution around* $\boldsymbol{\tau}$, *then for any fixed matrix* $\mathbf{C}$, *the vectors* $\mathbf{X}_i = \mathbf{C}\mathbf{Y}_i$, $i = 1, \cdots, n$, *are iid symmetric around* $\boldsymbol{\theta} = \mathbf{C}\boldsymbol{\tau}$.

Proof. It is clear that the $\mathbf{X}_i$'s are iid since the $\mathbf{Y}_i$'s are. Moreover, since the $\mathbf{Y}_i$ are symmetric around $\boldsymbol{\tau}$, then

$$(\mathbf{Y}_i - \boldsymbol{\tau}) \overset{d}{=} -(\mathbf{Y}_i - \boldsymbol{\tau}).$$

Using the transformation $H(\mathbf{u}) = \mathbf{Cu}$ results in

$$\begin{aligned} \mathbf{X}_i - \boldsymbol{\theta} = \mathbf{C}\mathbf{Y}_i - \mathbf{C}\boldsymbol{\tau} = H(\mathbf{Y}_i - \boldsymbol{\tau}) &\overset{d}{=} H(\boldsymbol{\tau} - \mathbf{Y}_i) \\ &= \mathbf{C}\boldsymbol{\tau} - \mathbf{C}\mathbf{Y}_i \\ &= \boldsymbol{\theta} - \mathbf{X}_i. \end{aligned}$$

Thus, the $\mathbf{X}_i$'s are symmetric around $\boldsymbol{\theta}$. ☐

Note that, for the simple repeated-measures setting considered, $\boldsymbol{\tau} = \tau \mathbf{1}'$, $\mathbf{C}$ is given in (5.4), and $\boldsymbol{\theta} = \mathbf{0}$.

Lemma 5.3.2. *If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid from a directionally symmetric distribution, then for any fixed matrix $\mathbf{C}$, the vectors $\mathbf{X}_i = \mathbf{C}\mathbf{Y}_i$, $i = 1, \cdots, n$, are also iid from a directionally symmetric distribution.*

Proof. It is clear that the $\mathbf{X}_i$'s are iid since the $\mathbf{Y}_i$'s are. Moreover,

$$\frac{\mathbf{Y}_i}{\|\mathbf{Y}_i\|} \overset{d}{=} -\frac{\mathbf{Y}_i}{\|\mathbf{Y}_i\|},$$

since the $\mathbf{Y}_i$'s come from a directionally symmetric distribution. Using the transformation $H(\mathbf{u}) = \mathbf{C}\mathbf{u}$, results in

$$\frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{Y}_i\|} \overset{d}{=} -\frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{Y}_i\|}.$$

Applying the transformation $\psi(\mathbf{v}) = \mathbf{v}/\|\mathbf{v}\|, \|\mathbf{v}\| \neq 0$ gives

$$\begin{aligned}
\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|} = \frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{C}\mathbf{Y}_i\|} = \frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{Y}_i\|}\frac{\|\mathbf{Y}_i\|}{\|\mathbf{C}\mathbf{Y}_i\|} = \psi\left(\frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{Y}_i\|}\right) &\overset{d}{=} \psi\left(-\frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{Y}_i\|}\right) \\
&= -\frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{Y}_i\|}\frac{\|\mathbf{Y}_i\|}{\|\mathbf{C}\mathbf{Y}_i\|} \\
&= -\frac{\mathbf{C}\mathbf{Y}_i}{\|\mathbf{C}\mathbf{Y}_i\|} \\
&= -\frac{\mathbf{X}_i}{\|\mathbf{X}_i\|}.
\end{aligned}$$

Thus, the $\mathbf{X}_i$'s have a distribution that is directional symmetric. □

The null asymptotic distributions of S_n and $W_{n\ p-1}$ are discussed in the following two theorems.

Theorem 5.3.1. *If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ are iid from a directionally symmetric distribution, then, under H_o, the quadratic form, S_n, defined in (5.5) has limiting chi-square distribution with $p-1$ degrees of freedom.*

Proof. The proof follows readily from lemma 5.3.1 and theorem 1 of Randles (2000). ☐

Theorem 5.3.2. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from a symmetric distribution, then, under* H_o*, the quadratic form,* $W_{n\ p-1}$*, defined in (5.6) has limiting chi-square distribution with* $p-1$ *degrees of freedom.*

Proof. The proof follows readily from lemma 5.3.2 and theorem 2.4.1 in chapter 2. ☐

CHAPTER 6
TEST COMPARISONS

6.1 Asymptotic Distribution Under Contiguous Alternatives

In the first set of test comparisons, the test statistics S_n and $W_{n\ p-1}$, defined in (5.5) and (5.6) respectively, along with PR_n, which is the test of Peters and Randles applied to the transformed observations (see Jan and Randles, 1996), are compared using their corresponding Pitman asymptotic relative efficiencies relative to Hotelling-Hsu T^2.

The computation of the ARE's of these tests requires finding their corresponding asymptotic distributions under contiguous alternatives. These derivations are done assuming that the $\mathbf{Y}$'s come from the class of elliptically symmetric distributions. Thus, assume that the $\mathbf{Y}$'s come from a distribution of the form

$$f_{\boldsymbol{\tau},\boldsymbol{\Sigma}}(\mathbf{y}) = K_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}}h\{(\mathbf{y}-\boldsymbol{\tau})'\boldsymbol{\Sigma}^{-1}(\mathbf{y}-\boldsymbol{\tau})\}, \tag{6.1}$$

where $K_p > 0$, and $\boldsymbol{\Sigma}$ is proportional to the covariance matrix of $\mathbf{Y}$. The following lemma, which is given as an exercise in Fang, Kotz and Ng's (1990) book, is stated.

Theorem 6.1.1. *If* $\mathbf{Y}$ *has a p-variate elliptically symmetric distribution with parameters* $\boldsymbol{\tau}$ *and* $\boldsymbol{\Sigma}$*, then* $\mathbf{X} = \mathbf{CY}$*, defined in (5.3), has a* $(p-1)$*-variate elliptically symmetric distribution with parameters* $\mathbf{C}\boldsymbol{\tau}$ *and* $\mathbf{C}\boldsymbol{\Sigma}\mathbf{C}'$*.*

Proof. see Theorem 2.16 of Fang, Kotz, and Ng, 1990. ☐

Before investigating the asymptotic distributions of S_n and $W_{n\ p-1}$ under contiguous alternatives, the density function of the transformed observations, $\mathbf{X}$'s, as well as that of $R = \mathbf{X}'\mathbf{X}$ are obtained.

Theorem 6.1.2. *If* $\mathbf{Y}$ *has a density function given by (6.1), then* $\mathbf{X}$*, defined in (5.3), has a density function of the form*

$$f_{\mathbf{C}\boldsymbol{\tau},\mathbf{C}\boldsymbol{\Sigma}\mathbf{C}'}(\mathbf{X}) = K_p|\mathbf{C}\boldsymbol{\Sigma}\mathbf{C}'|^{-\frac{1}{2}}h^*\{(\mathbf{X}-\mathbf{C}\boldsymbol{\tau})'(\mathbf{C}\boldsymbol{\Sigma}\mathbf{C}')^{-1}(\mathbf{X}-\mathbf{C}\boldsymbol{\tau})\},$$

where

$$h^*(t) = \int_{-\infty}^{+\infty} h(t+s^2)ds. \tag{6.2}$$

Proof. This proof is the same as the proof given by Jan (1991) but for the general case of all elliptically symmetric distribution. Consider the transformation,

$$\begin{aligned}
\mathbf{z} = \begin{bmatrix} \mathbf{x} \\ y_p \end{bmatrix} &= \begin{bmatrix} y_1 - y_p \\ \vdots \\ y_{p-1} - y_p \\ y_p \end{bmatrix} \\
&= \begin{bmatrix} 1 & \cdots & -1 \\ & \ddots & -1 \\ & & 1 \end{bmatrix} \mathbf{y} \\
&= \begin{bmatrix} \mathbf{C} \\ \mathbf{e}_p' \end{bmatrix} \mathbf{y} \\
&= \mathbf{B}\mathbf{y},
\end{aligned}$$

where $\mathbf{C}$ is given in (5.4), and $\mathbf{e}_p = (0,\cdots,0,1)'$. Note that $\mathbf{B}$ is nonsingular and $|\mathbf{B}| = 1$, thus $\mathbf{y} = \mathbf{B}^{-1}\mathbf{z}$, and

$$\begin{aligned}
f_{\mathbf{Z}}(\mathbf{z}) &= f_{\mathbf{y}}(\mathbf{y})|_{\mathbf{y}=\mathbf{B}^{-1}\mathbf{z}} \\
&= K_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}}h\{(\mathbf{B}^{-1}\mathbf{z}-\boldsymbol{\tau})'\boldsymbol{\Sigma}^{-1}(\mathbf{B}^{-1}\mathbf{z}-\boldsymbol{\tau})\} \\
&= K_p|\boldsymbol{\Sigma}|^{-\frac{1}{2}}h\{(\mathbf{z}-\mathbf{B}\boldsymbol{\tau})'(\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')^{-1}(\mathbf{z}-\mathbf{B}\boldsymbol{\tau})\} \\
&= K_p|\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}'|^{-\frac{1}{2}}h\{(\mathbf{z}-\mathbf{B}\boldsymbol{\tau})'(\mathbf{B}\boldsymbol{\Sigma}\mathbf{B}')^{-1}(\mathbf{z}-\mathbf{B}\boldsymbol{\tau})\}.
\end{aligned}$$

The last equality is due to the fact that $|\mathbf{B\Sigma B'}| = |\mathbf{B}||\mathbf{\Sigma}||\mathbf{B'}| = |\mathbf{\Sigma}|$ since $|\mathbf{B}| = 1$.

Now, the density function of $\mathbf{X}$ is

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) &= \int_{-\infty}^{+\infty} f_{\mathbf{Z}}(\mathbf{z}) dy_p \\ &= K_p|\mathbf{B\Sigma B'}|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} h\{(\mathbf{z} - \mathbf{B}\boldsymbol{\tau})'(\mathbf{B\Sigma B'})^{-1}(\mathbf{z} - \mathbf{B}\boldsymbol{\tau})\} dy_p \\ &= K_p|\mathbf{\Lambda}|^{-\frac{1}{2}} \int_{-\infty}^{+\infty} h\{(\mathbf{z} - \boldsymbol{\mu})'(\mathbf{\Lambda})^{-1}(\mathbf{z} - \boldsymbol{\mu})\} dy_p, \end{aligned} \tag{6.3}$$

where

$$\mathbf{\Lambda} = \mathbf{B\Sigma B'} \quad \text{and} \quad \boldsymbol{\mu} = \mathbf{B}\boldsymbol{\tau}.$$

The matrix $\mathbf{\Lambda}$ and the vector $\boldsymbol{\mu}$ can be also written as

$$\mathbf{\Lambda} = \mathbf{B\Sigma B'} = \begin{bmatrix} \mathbf{C\Sigma C'} & \mathbf{C\Sigma e}_p \\ \mathbf{e}_p'\mathbf{\Sigma C'} & \mathbf{e}_p'\mathbf{\Sigma e}_p \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} \\ \mathbf{\Lambda}_{21} & \Lambda_{22} \end{bmatrix},$$

and

$$\boldsymbol{\mu} = \mathbf{B}\boldsymbol{\tau} = \begin{bmatrix} \mathbf{C}\boldsymbol{\tau} \\ \mathbf{e}_p'\boldsymbol{\tau} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta} \\ \tau_p \end{bmatrix}.$$

The inverse of $\mathbf{\Lambda}$ has the following form:

$$\begin{aligned} \mathbf{\Lambda}^{-1} &= \begin{bmatrix} \mathbf{\Lambda}_{11} & \mathbf{\Lambda}_{12} \\ \mathbf{\Lambda}_{21} & \Lambda_{22} \end{bmatrix}^{-1} \\ &= \begin{bmatrix} \mathbf{\Lambda}_{11}^{-1} + \mathbf{\Lambda}_{11}^{-1}\mathbf{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1}\mathbf{\Lambda}_{21}\mathbf{\Lambda}_{11}^{-1} & -\mathbf{\Lambda}_{11}^{-1}\mathbf{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1} \\ -\Lambda_{22\cdot 1}^{-1}\mathbf{\Lambda}_{12}\mathbf{\Lambda}_{11}^{-1} & \Lambda_{22\cdot 1}^{-1} \end{bmatrix}, \end{aligned}$$

where

$$\Lambda_{22\cdot 1} = \Lambda_{22} - \mathbf{\Lambda}_{21}\mathbf{\Lambda}_{11}^{-1}\mathbf{\Lambda}_{12}.$$

Also, the determinant of $\boldsymbol{\Lambda}$ is such that

$$|\boldsymbol{\Lambda}| = |\boldsymbol{\Lambda}_{11}| \cdot |\Lambda_{22} - \boldsymbol{\Lambda}_{21}\boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}| = |\boldsymbol{\Lambda}_{11}| \cdot |\Lambda_{22\cdot 1}|.$$

Thus, the quadratic form in (6.3) can be written as

$$\begin{aligned}
&(\mathbf{z} - \boldsymbol{\mu})'\boldsymbol{\Lambda}^{-1}(\mathbf{z} - \boldsymbol{\mu}) \\
&= (\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}(\mathbf{x} - \boldsymbol{\theta}) + (\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1}\boldsymbol{\Lambda}_{21}\boldsymbol{\Lambda}_{11}^{-1}(\mathbf{x} - \boldsymbol{\theta}) \\
&+ (y_p - \tau_p)^2\Lambda_{22\cdot 1}^{-1} - 2(\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1}(y_p - \tau_p) \\
&= (\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}(\mathbf{x} - \boldsymbol{\theta}) + [(y_p - \tau_p)\Lambda_{22\cdot 1}^{-1/2} - (\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1/2}]^2,
\end{aligned}$$

and

$$\begin{aligned}
&f_{\mathbf{X}}(\mathbf{x}) = K_p|\boldsymbol{\Lambda}_{11}|^{-1/2} \cdot |\Lambda_{22\cdot 1}|^{-1/2} \cdot \\
&\int_{-\infty}^{+\infty} h\left\{(\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}(\mathbf{x} - \boldsymbol{\theta}) + [(y_p - \tau_p)\Lambda_{22\cdot 1}^{-1/2} - (\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1/2}]^2\right\} dy_p
\end{aligned} \tag{6.4}$$

Taking $s = (y_p - \tau_p)\Lambda_{22\cdot 1}^{-1/2} - (\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}\boldsymbol{\Lambda}_{12}\Lambda_{22\cdot 1}^{-1/2}$, expression (6.4) becomes

$$f_{\mathbf{X}}(\mathbf{x}) = K_p|\boldsymbol{\Lambda}_{11}|^{-1/2} \cdot \int_{-\infty}^{+\infty} h\left\{(\mathbf{x} - \boldsymbol{\theta})'\boldsymbol{\Lambda}_{11}^{-1}(\mathbf{x} - \boldsymbol{\theta}) + s^2\right\} ds.$$

This completes the proof, since $\boldsymbol{\Lambda}_{11} = \mathbf{C\Sigma C}'$ and $\boldsymbol{\theta} = \mathbf{C}\boldsymbol{\tau}$. □

Corollary 6.1.1. *If* $\mathbf{Y}$ *has a density function given by (6.1), and* $\mathbf{C\Sigma C}' = \mathbf{I}_{p-1}$, *then, under* H_o, $R = \mathbf{X}'\mathbf{X}$ *has a density function of the form*

$$g_R(r) = \frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})} K_p r^{p-2} h^*(r^2), \;\; 0 \le r \le \infty,$$

where $h^*(\cdot)$ *is as defined in (6.2).*

Proof. The proof is a direct application of theorem 2.9, p. 35 of Fang, Kotz and Ng's 1990 book. (Also, see Hössjer and Croux, 1995.) □

Now, consider the sequence of alternatives where $\mathbf{X}_1, \cdots, \mathbf{X}_n$ are iid with density $f_{\mathbf{X}}(\mathbf{x} - \frac{\mathbf{c}}{\sqrt{n}})$, where $f_{\mathbf{x}}(\cdot)$ is given in (6.4), and $\mathbf{c}$ is a non-zero $(p-1)$-dimensional vector.

The following two theorems characterize the limiting distribution of $W_{n\ p-1}$ and S_n under the sequence of alternatives approaching the null.

Theorem 6.1.3. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from an elliptically symmetric distribution and* $\mathbf{C\Sigma C'} = \mathbf{I}_{p-1}$*, then, under the sequence of alternatives approaching the null hypothesis, the limiting distribution of* $W_{n\ p-1}$ *is non-central chi-square with* $p-1$ *degrees of freedom and non-centrality parameter*

$$\delta_{W_{n\ p-1}} = (\frac{4}{p-1})\frac{E^2_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right]}{E_{H_o}[\phi^2(G(R))]}\mathbf{c}'\mathbf{c}, \tag{6.5}$$

where $R_i^2 = \mathbf{X}'\mathbf{X}$, $h^{*'}(R^2) = \frac{\partial h^*(t)}{\partial t}|_{t=R^2}$, *and* $G(t) = P_{H_o}(R^2 \leq t),\ t \geq 0$.

Proof. When $\mathbf{C\Sigma C'} = \mathbf{I}_{p-1}$ and under the null hypothesis, the transformed observations are iid from an elliptically symmetric distribution with a density function of the form

$$f_{\mathbf{X}}(\mathbf{x}) = K_p h^*(\mathbf{x}'\mathbf{x}).$$

Therefore, the proof follows from applying theorem 3.1.1 to the transformed sample. □

Theorem 6.1.4. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from an elliptically symmetric distribution and* $\mathbf{C\Sigma C'} = \mathbf{I}_{p-1}$*, then under the sequence of alternatives approaching the null hypothesis, the limiting distribution of* S_n *is in non-central chi-square with* $p-1$ *degrees of freedom and non-centrality parameter*

$$\delta_{S_n} = (\frac{4}{p-1})E^2_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\right]\mathbf{c}'\mathbf{c}, \tag{6.6}$$

where $R^2 = \mathbf{X}'\mathbf{X}$ *and* $h^{*'}(R^2) = \frac{\partial h^*(t)}{\partial t}|_{t=R^2}$, *and* $G(t) = P_{H_o}(R^2 \leq t),\ t \geq 0$.

Proof. The proof follows from the fact that S_n can be obtained from $W_{n\ p-1}$ by setting $\phi(u) \equiv 1$. ☐

Note that the assumption $\mathbf{C\Sigma C}' = \mathbf{I}_{p-1}$ is used without loss of generality since the test statistics involved are affine-invariant.

6.2 Asymptotic Relative Efficiencies

The Pitman Asymptotic Relative Efficiency between two test statistics is defined in Chapter 3. In this section, the ARE of $W_{n\ p-1}$ and that of S_n, relative to Hotelling-Hsu T^2 are obtained. Two special families from the class of elliptically symmetric distributions are considered, the power family and the multivariate t-distributions family. These two families are described in Chapter 3. As in Chapter 3, the efficiencies obtained are ratios of non-centrality parameters of chi-square limiting distributions.

Theorem 6.2.1. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from the power family of distributions, then the Pitman asymptotic relative efficiency of* $W_{n\ p-1}$ *with respect to Hotelling-Hsu* T^2 *is*

$$ARE(W_{n\ p-1}, T^2) = (\frac{4}{p-1})\frac{E^2_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right]}{E_{H_o}[\phi^2(G(R))]}. \tag{6.7}$$

where

$$\begin{aligned} E_{H_o}&\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right] = \\ -K_p&\frac{2\nu\pi^{\frac{p-1}{2}}}{c_o^{\nu}\Gamma(\frac{p-1}{2})} \cdot \int_0^{\infty} \phi(G(r))r^{p-1}\left[\int_{-\infty}^{+\infty}(r^2+s^2)^{\nu-1}e^{-(\frac{r^2+s^2}{c_o})^{\nu}}ds\right]dr, \end{aligned} \tag{6.8}$$

and

$$G(r) = K_p\frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}\int_0^r\int_{-\infty}^{+\infty} t^{p-2}e^{-(\frac{t^2+s^2}{c_o})^{\nu}}ds\ dt. \tag{6.9}$$

Proof. Without loss of generality, let $\mathbf{C\Sigma C}' = \mathbf{I}_{p-1}$. Result (6.7) follows immediately from Theorem 6.1.3 and Theorem 3.2.1. To prove (6.8), recall that for the power family,

$$h(t) = \exp\{-(\frac{t}{c_o})^\nu\}, \text{ and, } K_p = \frac{\nu\Gamma(p/2)}{\Gamma(p/(2\nu))[\pi c_o]^{p/2}},$$

where c_o is given in (3.3), thus $h^*(t)$, defined in (6.2), is such that

$$h^*(t) = \int_{-\infty}^{+\infty} \exp\{-(\frac{t+s^2}{c_o})^\nu\}ds,$$

and

$$\frac{\partial h^*(t)}{\partial t} = -\frac{\nu}{c_o^\nu}\int_{-\infty}^{+\infty} (t+s^2)^{\nu-1}e^{-(\frac{t+s^2}{c_o})^\nu}ds.$$

(see Rudin, 1976, Theorem 9.42, p.236). Moreover, by corollary 6.1.1,

$$g_R(r) = K_p\frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}r^{p-2}h^*(r^2),$$

thus

$$\begin{aligned} E_{H_o}&\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right] \\ &= \int_0^{+\infty}\left[\phi(G(r))r\frac{h^{*'}(r^2)}{h^*(r^2)}\right]g_R(r)dr \\ &= \frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}K_p\int_0^{+\infty}\phi(G(r))r^{p-1}h^{*'}(r^2)dr. \end{aligned}$$

Replacing $h^{*'}(r^2)$ by its corresponding expression, (6.8) is established. Expression (6.9) is easily established by replacing $g(\cdot)$ by its corresponding expression in

$$G(r) = \int_0^r g(t)dt$$

□

Theorem 6.2.2. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from the power family of distributions, then the Pitman asymptotic relative efficiency of* S_n *with respect to Hotelling-Hsu* T^2 *is*

$$ARE(S_n, T^2) = (\frac{4}{p-1})E_{H_o}^2\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\right], \tag{6.10}$$

where

$$E_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\right] = -K_p\frac{2\nu\pi^{\frac{p-1}{2}}}{c_o^{\nu}\Gamma(\frac{p-1}{2})}\cdot\int_0^{\infty} r^{p-1}\left[\int_{-\infty}^{+\infty}(r^2+s^2)^{\nu-1}e^{-(\frac{r^2+s^2}{c_o})^{\nu}}ds\right]dr.$$

Proof. Without loss of generality, let $\mathbf{C\Sigma C}' = \mathbf{I}_{p-1}$. Thus, the proof follows by setting $\phi(u)\equiv 1$ when applying theorem 6.2.1. $\square$

Theorem 6.2.3. *If* $\mathbf{Y}_1,\cdots,\mathbf{Y}_n$ *are iid from the family of multivariate t-distributions, then the Pitman asymptotic relative efficiency of* $W_{n\ p-1}$ *with respect to Hotelling-Hsu* T^2 *is*

$$ARE(W_{n\ p-1},T^2) = (\frac{4\nu}{(p-1)(\nu-1)})\frac{E^2_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right]}{E_{H_o}[\phi^2(G(R))]}, \tag{6.11}$$

where

$$E_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right] = \\ -K_p\frac{(p+\nu)\pi^{\frac{p-1}{2}}}{\nu\Gamma(\frac{p-1}{2})}\cdot\int_0^{\infty}\phi(G(r))r^{p-1}\left[\int_{-\infty}^{+\infty}\left(1+\frac{r^2+s^2}{\nu}\right)^{-\frac{p+\nu+2}{2}}ds\right]dr, \tag{6.12}$$

and

$$G(r) = K_p\frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})}\int_0^{r}\int_{-\infty}^{+\infty}t^{p-2}\left(1+\frac{t^2+s^2}{\nu}\right)^{-\frac{p+\nu}{2}}ds\ dt. \tag{6.13}$$

Proof. Without loss of generality, let $\mathbf{C\Sigma C}' = \mathbf{I}_{p-1}$. Expression (6.11) follows from theorem 6.1.3 and theorem 3.2.2. To establish (6.12), recall that for multivariate t-distributions,

$$h(t) = (1+\frac{t}{\nu})^{-\frac{p+\nu}{2}},\ \text{and},\ K_p = \frac{\Gamma((p+\nu)/2)}{\Gamma(\nu/2))[\pi\nu]^{p/2}}.$$

Therefore, $h^*(t)$, defined in (6.2), is such that

$$h^*(t) = \int_{-\infty}^{+\infty}\left(1+\frac{t+s^2}{\nu}\right)^{-\frac{p+\nu}{2}}ds,$$

and

$$\frac{\partial h^*(t)}{\partial t} = -\frac{p+\nu}{2\nu}\int_{-\infty}^{+\infty}\left(1+\frac{t+s^2}{\nu}\right)^{-\frac{p+\nu+2}{2}}ds.$$

The density function of R becomes

$$g_R(r) = K_p \frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})} r^{p-2} \int_{-\infty}^{+\infty} \left(1 + \frac{t+s^2}{\nu}\right)^{-\frac{p+\nu}{2}} ds. \tag{6.14}$$

Hence,

$$E_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\phi(G(R))\right] = K_p \frac{2\pi^{\frac{p-1}{2}}}{\Gamma(\frac{p-1}{2})} \int_0^{+\infty} \phi(G(r)) r^{p-1} h^{*'}(r^2) dr,$$

which, by replacing $h^{*'}(r^2)$ by its value, gives (6.12). Finally, (6.13) is obtained by integrating expression (6.14) over $[0, r]$. □

Theorem 6.2.4. *If* $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ *are iid from the family of multivariate t-distributions, then the Pitman asymptotic relative efficiency of* S_n *with respect to Hotelling-Hsu* T^2 *is*

$$ARE(S_n, T^2) = (\frac{4\nu}{(p-1)(\nu-1)}) E^2_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\right], \tag{6.15}$$

where

$$E_{H_o}\left[R\frac{h^{*'}(R^2)}{h^*(R^2)}\right] = -K_p \frac{(p+\nu)\pi^{\frac{p-1}{2}}}{\nu\Gamma(\frac{p-1}{2})} \cdot \int_0^{\infty} r^{p-1} \left[\int_{-\infty}^{+\infty} \left(1 + \frac{r^2+s^2}{\nu}\right)^{-\frac{p+\nu+2}{2}} ds\right] dr.$$

Proof. This theorem is a special case of theorem 6.2.3, where $\phi(\cdot) \equiv 1$. □

The asymptotic relative efficiencies of the tests obtained by applying the new signed-rank test with the linear $\phi(\cdot)$ function, W_{lnp}, the sign test of Randles (2000), S_n, and the signed-rank test of Peters and Randles (1990), PR_n, to the repeated-measure designs are presented in tables 6.1 and 6.2. Table 6.1 displays the efficiencies for the cases when the observations come from the power family of distributions, while Table 6.2 holds the results for the cases when the observations come from the family of multivariate t-distributions.

Table 6.1: Asymptotic Relative Efficiencies Under the Power Family of Distributions for the Repeated-Measures Problem

		ν				
Dimension	Test	5	2	1	.5	.1
2	$W_{ln\ p-1}$	.876	.878	.955	1.266	
	PR_n	.876	.878	.955	1.266	
	S_n	.420	.478	.637	1.216	
3	$W_{ln\ p-1}$	.943	.927	.971	1.206	
	PR_n	1.083	1.012	.985	1.078	8.410
	S_n	.574	.636	.785	1.234	50.710
4	$W_{ln\ p-1}$	.973	.946	.978	1.167	11.269
	PR_n	1.181	1.054	.975	.987	4.202
	S_n	.660	.719	.849	1.201	20.173
5	$W_{ln\ p-1}$	.989	.957	.982	1.140	7.418
	PR_n	1.231	1.066	.961	.935	2.786
	S_n	.716	.771	.884	1.171	11.424
8	$W_{ln\ p-1}$	1.012	.976	.989	1.092	3.694
	PR_n	1.273	1.059	.929	.861	1.517
	S_n	.810	.852	.931	1.115	4.741
20	$W_{ln\ p-1}$	1.025	.996	.997	1.037	1.707
	PR_n	1.201	.990	.871	.792	.856
	S_n	.919	.939	.974	1.049	1.893

Entries: Pitman asymptotic relative efficiencies with respect to Hotelling-Hsu's T^2.
Distribution = Power family.
$W_{ln\ p-1}$ = New procedure, with the linear $\phi(\cdot)$, applied to the **X**'s.
PR_n = Peters and Randles procedure applied to the **X**'s.
S_n = Randles (2000) procedure applied to the **X**'s.

Table 6.2: Asymptotic Relative Efficiencies Under the Multivariate t-Distributions for the Repeated-Measures Problem

		ν			
Dimension	Test	3	6	10	20
2	$W_{ln\ p-1}$	1.900	1.164	1.054	.997
	PR_n	1.900	1.164	1.054	.997
	S_n	1.621	.879	.757	.690
3	$W_{ln\ p-1}$	1.954	1.813	1.068	1.011
	PR_n	1.041	1.123	1.005	.985
	S_n	2.000	1.084	.934	.851
4	$W_{ln\ p-1}$	2.010	1.196	1.077	1.018
	PR_n	1.621	1.064	1.000	.978
	S_n	2.162	1.172	1.009	.920
5	$W_{ln\ p-1}$	2.055	1.209	1.084	1.023
	PR_n	1.533	1.018	.964	.952
	S_n	2.250	1.220	1.051	.958
8	$W_{ln\ p-1}$	2.143	1.235	1.099	1.032
	PR_n	1.386	.930	.892	.894
	S_n	2.371	1.286	1.107	1.009
20	$W_{ln\ p-1}$	2.270	1.275	1.121	1.043
	PR_n	1.225	.816	.785	.797
	S_n	2.480	1.345	1.158	1.056

Entries: Pitman asymptotic relative efficiencies with respect to Hotelling-Hsu's T^2.
Distribution = t-distribution.
$W_{ln\ p-1}$ = New procedure, with the linear $\phi(\cdot)$, applied to the $\mathbf{X}$'s.
PR_n = Peters and Randles procedure applied to the $\mathbf{X}$'s.
S_n = Randles (2000) procedure applied to the $\mathbf{X}$'s.

Examining Table 6.1, it is clear that $W_{ln\ p-1}$ has good efficiencies over a wide spectrum of distributions, ranging from the multivariate normal case ($\nu = 1$) to heavy-tailed cases ($\nu = .1,\ .5$) and light-tailed cases ($\nu = 2,\ 5$). Its efficiencies are always close to those of Hotelling-Hsu and in heavy-tailed cases they are considerably better. Moreover, these strong efficiencies are stable as the dimension increases, unlike the efficiencies of PR_n. For multivariate normal distributions, $W_{ln\ p-1}$ is virtually as good as Hotelling-Hsu's T^2, while PR_n and S_n have good efficiencies only for small dimensions and large dimensions, respectively. For the heavy-tailed cases, S_n and $W_{ln\ p-1}$ have efficiencies that are always higher than one while those of PR_n are only so for the cases $p = 2$ and $p = 3$. For the light-tailed distributions, the efficiencies of $W_{ln\ p-1}$, and PR_n are significantly higher than those of S_n, with PR_n being the best. As the dimension increases to ∞, the efficiencies of $W_{ln\ p-1}$ and S_n approach one but those of PR_n approach .75. Note that, the empty cell in this table is caused by the fact that when $4\nu + p \leq 3$, the efficiencies do not exist (see Jan and Randles, 1996).

In Table 6.2, four different multivariate t-distributions are considered. These are the multivariate t-distributions with degrees of freedom 3, 6, 10, and 20. The results show that the efficiencies of $W_{ln\ p-1}$ and S_n are not only higher than those of PR_n but also those efficiencies are always higher than one, except for those of S_n when the dimension is low and the degrees of freedom are high. Again we see the performance of PR_n deteriorating as the dimension increases.

In summary, one can conclude that $W_{ln\ p-1}$ has strong efficiencies for all cases considered while those of S_n and PR_n are only strong for heavy-tailed and light-tailed distributions, respectively. The efficiencies also demonstrate that $W_{ln\ p-1}$ is more efficient than T^2 for heavy-tailed cases and virtually as good as T^2 for multivariate normal distributions and distributions with lighter tails.

6.3 Monte Carlo Results

To compare the performances of S_n and $W_{ln\ p-1}$ with respect to other procedures in the literature, a Monte Carlo study for dimension $p = 4$, sample size $n = 30$, and significance level $\alpha = .05$ was conducted. Besides S_n and $W_{ln\ p-1}$, the study included the ANOVA F test, Hotelling-Hsu T^2, Friedman's test (F_R), Iman, Hora, and Convor (1984) rank transformation version of the ANOVA F test (F_{RT}), and Quade's test (W_Q). Note that Quade's test was computed using the range as the location-free statistic that measures the variability within the components of each observed vector.

Three of the seven procedures considered are computed on the transformed observations, that is the **X**'s. These are T^2, S_n and $W_{nl\ p-1}$, while the rest of the procedures use the original data, the **Y**'s. The test statistics F, F_{RT}, and W_Q are compared to upper α-th quantile of an F distribution with $(p-1)$ and $(n-1)(p-1)$, numerator and denominator degrees of freedom, respectively. No adjustment for the degrees of freedom was made since we noticed that doing so will only correct the α level of the test but will not have any significant effect on the power (see Jan and Randles, 1996). On the other hand, the tests F_R, S_n and $W_{ln\ p-1}$ are compared to the upper α-th quantile of a chi-square distribution with $p-1$ degrees of freedom. The Hotelling-Hsu T^2 is compared to $(n-1)(p-1)/(n-p+1)$ times the upper α-th quantile of an F distribution with $(p-1)$ and $(n-p+1)$, numerator and denominator degrees of freedom, respectively.

The test statistics were compared under elliptically symmetric distributions with densities from the power family and the family of multivariate t-distributions. From the power family, the heavy-tailed case ($\nu = .1$), the multivariate normal case ($\nu = 1$), and the light-tailed case ($\nu = 50$) were used. From the family of multivariate t-distributions, distributions with degrees of freedom of 1, 3, and 10 were considered. For test statistics that use the original observations, the distributions were located at $\boldsymbol{\tau} = (k\delta, k\delta, k\delta, 0)$, $k = 0, 1, 2, 3$, and for the rest of the tests, the distributions were located at $\boldsymbol{\theta} = (k\delta, k\delta, k\delta)$, $k = 0, 1, 2, 3$. The value δ was adjusted for different distributions to obtain somewhat similar points on the power curve. Two different types of the population variance-covariance matrix were considered, the first, which satisfies Huynh and Feldt condition, is $\mathbf{\Sigma} = \mathbf{I}_4$, the identity matrix, the other, which does not satisfy the Huynh and Feldt condition is

$$\mathbf{\Sigma} = \mathbf{E}\mathbf{E}' = \begin{bmatrix} 14 & -1 & -2 & 16 \\ -1 & 22 & 1 & 1 \\ -2 & 1 & 3 & -7 \\ 16 & 1 & -7 & 30 \end{bmatrix}. \tag{6.16}$$

This was done in order to show the dependence of the performances of F, F_R, and F_{RT} on the type of the population variance-covariance matrix. Tables 6.3 and 6.4 contain the results from the Monte Carlo study done for distributions from the power family, while Tables 6.5 and 6.6 contain those for the distributions from the family of t-distributions. Entries in each table are the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of the corresponding null distribution.

For the affine-invariant test statistics considered, that is, T^2, S_n, and $W_{ln\ p-1}$, the results from the Monte Carlo studies agree with those obtained from the efficiency comparisons. Thus, for the multivariate normal distribution, it is clear that the three tests behave very similarly, with T^2 having the highest power. For the heavy-tailed distribution from the power family ($\nu = .1$), the performance of T^2 deteriorates dramatically, while those of S_n and $W_{ln\ p-1}$ are still strong with S_n being the better of the two. For the light-tailed case of the power family ($\nu = 50$), the power of S_n is clearly smaller than the other two statistics. Moreover, for the multivariate t-distributions, the performances of the three statistics became very similar as the degrees of freedom increased. While for small degrees of freedom,

S_n has the strongest performance followed by a good performance from $W_{ln\ p-1}$, while that of T^2 deteriorates significantly. As expected, these performances did not change when the structure of the populations variance-covariance matrix changed from $\mathbf{I}_4$ to $\mathbf{EE}'$. This can be seen by comparing table 6.3 with Table 6.4 and Table 6.5 with Table 6.6.

Comparing all seven tests, the following observations are made. When the underlying distribution is multivariate normal with a variance-covariance matrix that satisfies Huynh and Feldt condition ($\mathbf{\Sigma} = \mathbf{I}_4$), all seven tests showed a good performance, with F being the best (see, Table 6.3, case $\nu = 1$). However, when the variance-covariance matrix did not satisfy Huynh and Feldt condition ($\mathbf{\Sigma} = \mathbf{EE}'$), only the affine-invariant tests maintained a strong performance, while all the other four tests, showed a significant decrease in power and they didn't achieve the designed .05 α-level of the test (see, Table 6.4, case $\nu = 1$). For the heavy-tailed distribution from the power family, with $\mathbf{\Sigma} = \mathbf{I}_4$, S_n and F_R showed the highest powers, followed by F_{RT}, $W_{ln\ p-1}$, and W_Q, with a good performance, while T^2, and F didn't performance well (see, Table 6.3, case $\nu = .1$). On the other hand, when $\mathbf{\Sigma} = \mathbf{EE}'$, only S_n kept a strong performance, the performance of F_R decreased to become comparable to that of $W_{ln\ p-1}$, while the rest of the tests performed poorly (see, Table 6.4, case $\nu = .1$). As for the light-tailed distribution from the power family, with $\mathbf{\Sigma} = \mathbf{I}_4$, W_Q, and F performed the best, followed by good performances from T^2, F_{RT}, and $W_{ln\ p-1}$, while S_n and F_R performed poorly (see, Table 6.3, case $\nu = 50$). However, when $\mathbf{\Sigma} = \mathbf{EE}'$, T^2, and $W_{ln\ p-1}$ kept the strongest performances with S_n performing well, while all other tests performed poorly (see, Table 6.4, case $\nu = 50$).

Now, results from Monte Carlos studies for the family of t-distributions are presented. For the multivariate t-distribution with 1 degree of freedom and $\mathbf{\Sigma} = \mathbf{I}_4$, F_R, and S_n performed the best, followed by good performances from F_{RT}, and

$W_{ln\ p-1}$, while the rest of the tests performed poorly, especially F, and T^2 (see, Table 6.5, case df=1). On the other hand, when $\mathbf{\Sigma} = \mathbf{EE}'$, the performance of F_{RT} dropped to below that of $W_{ln\ p-1}$ with S_n still being the best and all the other tests performing poorly (see, Table 6.6, case df=1). For the multivariate t-distribution with 3 degrees of freedom and $\mathbf{\Sigma} = \mathbf{I}_4$, all seven tests performed well, with F_{RT}, F_R, and S_n being the best (see, Table 6.5, case df=3). However, when $\mathbf{\Sigma} = \mathbf{EE}'$, only S_n, $W_{ln\ p-1}$, and T^2 maintained a strong performance, with a slight advantage of S_n and $W_{ln\ p-1}$ over T^2 (see, Table 6.6, case df=3). Finally, for the multivariate t-distribution with 10 degrees of freedom and $\mathbf{\Sigma} = \mathbf{I}_4$, all seven tests have virtually the same performance, with F_{RT} the highest power (see, Table 6.5, case df=10). On the other hand, when $\mathbf{\Sigma} = \mathbf{EE}'$, only T^2, S_n and $W_{ln\ p-1}$ maintained a strong performance, while the performances of the other tests deteriorated.

In summary, the dependence of the tests F, F_{RT}, F_R, and W_Q on the structure of the variance-covariance matrix of the underlying population was clear under all different distributional assumptions. The effect was obvious as the powers of these tests deteriorated and the sizes of the tests shifted grossly from the .05 α-level. For heavy-tailed distributions, S_n, was the best overall performance, with $W_{ln\ p-1}$ and F_R having good performances. While for the light-tailed distributions $W_{ln\ p-1}$ and T^2 were always good choices followed by S_n. Thus, one can conclude that the performance of $W_{ln\ p-1}$ was good for all settingS studied.

Table 6.3: Monte Carlo Results for the Power Family, with $\mathbf{\Sigma}$=$\mathbf{I}_4$

Test Statistic	ν	Amount of Shift			
	.1	0	1Δ	2Δ	3Δ
T^2		0.0233	0.0620	0.1953	0.4100
S_n		0.0387	0.2663	0.6740	0.8960
$W_{ln\ p-1}$		0.0417	0.1600	0.4443	0.6830
F		0.0867	0.0553	0.1517	0.3210
F_{RT}		0.0477	0.1567	0.4957	0.7740
F_R		0.0420	0.2747	0.6777	0.8987
W_Q		0.0427	0.1070	0.3050	0.5597
	1				
T^2		0.0500	0.1353	0.4277	0.7813
S_n		0.0437	0.1150	0.3603	0.7173
$W_{ln\ p-1}$		0.0470	0.1203	0.3790	0.7483
F		0.0430	0.1443	0.4703	0.8300
F_{RT}		0.0477	0.1403	0.4570	0.8100
F_R		0.0420	0.1060	0.3613	0.7120
W_Q		0.0413	0.1233	0.4030	0.7727
	50				
T^2		0.0510	0.1360	0.4077	0.7847
S_n		0.0477	0.0963	0.2700	0.5903
$W_{ln\ p-1}$		0.0480	0.1193	0.3600	0.7337
F		0.0473	0.1443	0.4617	0.8403
F_{RT}		0.0510	0.1310	0.4060	0.7773
F_R		0.0493	0.1030	0.2857	0.5850
W_Q		0.0493	0.1470	0.4753	0.8523

Entries: the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Power Family.
α = .05.
$\mathbf{\Sigma}$ = $\mathbf{I}_4$.
Sample size = 30.
Dimension = 4.
T^2 = Hotelling-Hsu procedure.
S_n = Randles (2000) procedure.
$W_{ln\ p-1}$ = New procedure with the linear $\phi(\cdot)$ function.
F = ANOVA F Test.
F_{RT} = Rank Transformation ANOVA Test.
F_R = Friedman's Test.
W_Q = Quade's Test.

Table 6.4: Monte Carlo Results for the Power Family, with $\mathbf{\Sigma}=\mathbf{EE}'$

Test Statistic	ν	Amount of Shift			
	.1	0	1Δ	2Δ	3Δ
T^2		0.0273	0.0697	0.2157	0.4350
S_n		0.0440	0.2750	0.6827	0.9087
$W_{ln\ p-1}$		0.0433	0.1700	0.4487	0.6997
F		0.0517	0.0670	0.0977	0.1587
F_{RT}		0.0707	0.1087	0.2217	0.4017
F_R		0.0530	0.1570	0.4020	0.6697
W_Q		0.0650	0.0983	0.1887	0.3080
	1				
T^2		0.0470	0.1380	0.4663	0.8463
S_n		0.0433	0.1143	0.3910	0.7640
$W_{ln\ p-1}$		0.0423	0.1210	0.4150	0.8047
F		0.0670	0.1017	0.2147	0.4027
F_{RT}		0.0817	0.1073	0.1840	0.3227
F_R		0.0617	0.0910	0.1833	0.3823
W_Q		0.0690	0.1003	0.1990	0.3770
	50				
T^2		0.0533	0.1553	0.5060	0.8923
S_n		0.0433	0.1120	0.3410	0.7157
$W_{ln\ p-1}$		0.0443	0.1353	0.4543	0.8540
F		0.0743	0.1077	0.2330	0.4430
F_{RT}		0.0843	0.1037	0.1817	0.3223
F_R		0.0570	0.0837	0.1703	0.3377
W_Q		0.0773	0.1130	0.2317	0.4227

Entries: the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Power Family.
α = .05.
$\mathbf{\Sigma}$ = $\mathbf{EE}'$.
Sample size = 30.
Dimension = 4.
T^2 = Hotelling-Hsu procedure.
S_n = Randles (2000) procedure.
$W_{ln\ p-1}$ = New procedure with the linear $\phi(\cdot)$ function.
F = ANOVA F Test.
F_{RT} = Rank Transformation ANOVA Test.
F_R = Friedman's Test.
W_Q = Quade's Test.

Table 6.5: Monte Carlo Results for the Family of t-Distributions, with $\mathbf{\Sigma}$=$\mathbf{I}_4$

Test Statistic	df	Amount of Shift			
	1	0	1Δ	2Δ	3Δ
T^2		0.0183	0.0337	0.1043	0.2250
S_n		0.0430	0.1413	0.4797	0.7750
$W_{ln\ p-1}$		0.0423	0.1100	0.3387	0.5873
F		0.0163	0.0237	0.0513	0.1000
F_{RT}		0.0510	0.1180	0.3677	0.6510
F_R		0.0507	0.1510	0.4987	0.8040
W_Q		0.0473	0.0977	0.2590	0.4713
	3				
T^2		0.0400	0.1263	0.4253	0.7477
S_n		0.0397	0.1507	0.5230	0.8603
$W_{ln\ p-1}$		0.0390	0.1397	0.4750	0.8040
F		0.0373	0.1180	0.3810	0.7267
F_{RT}		0.0443	0.1600	0.5440	0.8777
F_R		0.0403	0.1553	0.5443	0.8750
W_Q		0.0443	0.1213	0.4397	0.7650
	10				
T^2		0.0463	0.1150	0.3787	0.7150
S_n		0.0397	0.1060	0.3503	0.6830
$W_{ln\ p-1}$		0.0413	0.1133	0.3507	0.6897
F		0.0433	0.1200	0.4030	0.7490
F_{RT}		0.0493	0.1247	0.4207	0.7673
F_R		0.0463	0.1097	0.3607	0.6973
W_Q		0.0493	0.1093	0.3573	0.6903

Entries: the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Multivariate t-distribution.
α = .05.
$\mathbf{\Sigma}$ = $\mathbf{I}_4$.
Sample size = 30.
Dimension = 4.
T^2 = Hotelling-Hsu procedure
S_n = Randles (2000) procedure.
$W_{ln\ p-1}$ = New procedure with the linear $\phi(\cdot)$ function.
F = ANOVA F Test.
F_{RT} = Rank Transformation ANOVA Test.
F_R = Friedman's Test.
W_Q = Quade's Test.

Table 6.6: Monte Carlo Results for the Family of t-Distributions, with $\mathbf{\Sigma}$=$\mathbf{EE}'$

Test Statistic	df	Amount of Shift			
	1	0	1Δ	2Δ	3Δ
T^2		0.0177	0.0433	0.1580	0.3227
S_n		0.0397	0.1977	0.6210	0.8963
$W_{ln\ p-1}$		0.0387	0.1463	0.4380	0.7083
F		0.0347	0.0353	0.0440	0.0733
F_{RT}		0.0663	0.1003	0.1997	0.3533
F_R		0.0600	0.1167	0.3230	0.6090
W_Q		0.0613	0.0907	0.1753	0.3220
	3				
T^2		0.0423	0.1297	0.4483	0.7870
S_n		0.0407	0.1710	0.5630	0.8980
$W_{ln\ p-1}$		0.0417	0.1467	0.5013	0.8490
F		0.0647	0.0863	0.1700	0.3397
F_{RT}		0.0750	0.0970	0.2050	0.4093
F_R		0.0610	0.0953	0.2667	0.5523
W_Q		0.0610	0.0910	0.2077	0.4000
	10				
T^2		0.0407	0.2143	0.6940	0.9653
S_n		0.0410	0.1887	0.6510	0.9617
$W_{ln\ p-1}$		0.0390	0.1927	0.6620	0.9607
F		0.0717	0.1273	0.3153	0.6137
F_{RT}		0.0733	0.1173	0.2760	0.5470
F_R		0.0560	0.1117	0.3163	0.6573
W_Q		0.0633	0.1217	0.3103	0.5733

Entries: the proportion of times out of 3000 iterations in which each test statistic exceeded the upper α-percentile of its asymptotic null distribution.

Distribution = Multivariate t-distribution.
α = .05.
$\mathbf{\Sigma}$ = $\mathbf{EE}'$.
Sample size = 30.
Dimension = 4.
T^2 = Hotelling-Hsu procedure.
S_n = Randles (2000) procedure.
$W_{ln\ p-1}$ = New procedure with the linear $\phi(\cdot)$ function.
F = ANOVA F Test.
F_{RT} = Rank Transformation ANOVA Test.
F_R = Friedman's Test.
W_Q = Quade's Test.

CHAPTER 7
SUMMARY, CONCLUSIONS, AND FUTURE RESEARCH

7.1 Summary and Conclusions

In this work, the multivariate one-sample location problem was investigated. A class of affine-invariant test statistics was introduced. Its small-sample conditional distribution-free property was established. The asymptotic null distribution as well as the asymptotic distribution under a sequence of alternatives approaching the null were obtained under the minimal assumption of symmetry of the underlying distribution of the observations. Moreover, a particular member of this class was recommended. Comparisons between this member and several competitors in the literature were based on the values of the corresponding Pitman asymptotic relative efficiencies of these tests and on the performances of these tests as seen in several Monte Carlo studies.

Further, an application of this new class of test statistics to the simple repeated-measures designs was established. The Pitman asymptotic relative efficiencies of two particular members were evaluated and then compared to those of another test statistic previously proposed for these designs. More comprehensive comparisons between these two members and several leading competitors were done through Monte Carlo studies.

The comparisons presented show that in the one-sample location problem, the recommended statistic has strong efficiencies and performs very well over a wide spectrum of distributions, ranging from very light-tailed distributions to very heavy-tailed ones. In particular, these distributions were taken from two specific members of the class of elliptically symmetric distributions; the power family and the family of multivariate t-distributions. Moreover, the test's efficiencies and performance didn't deteriorate when the dimension was increased as is the case for some of the other tests in the literature. The best feature of this statistic, which is the ease with which it can be computed for data in any practical dimension. This computational advantage was seen through the difference in computation times of Monte Carlo studies when other test statistics were included or excluded.

In regard to the application to the simple repeated-measures designs, similar results were seen. The two new test statistics considered had strong efficiencies and performances. Moreover, these performances were not affected by the structure of the population variance-covariance matrix, as is the case for many of the test statistics often recommended for such designs.

In summary, one can conclude that the new statistic is not only efficient but also practical to use. Its affine-invariance property, minimal assumption requirement regarding the population underlying distribution, strong efficiencies, and ease of computation make it one of the best test statistics to use both for one-sample location problems and simple repeated-measures designs.

7.2 Future Research

In the future, I would like to investigate other members of this proposed class for improved efficiencies and performance. Also, other applications of this test statistic will be researched and its performance will be compared to other existing test statistics.

APPENDIX A
COMPUTING $\hat{\mathbf{A}}$

To compute $\hat{\mathbf{A}}$, an iterative procedure is used. It begins with

$$\mathbf{S}_0 = n^{-1} \sum_{i=1}^{n} \left(\frac{\mathbf{Y}_i}{||\mathbf{Y}_i||} \right) \left(\frac{\mathbf{Y}_i}{||\mathbf{Y}_i||} \right)',$$

and forms $\hat{\mathbf{A}}_0 = \text{Chol}(\mathbf{S}_0^{-1})$, where Chol($\mathbf{M}$) denotes the upper triangular Cholesky factorization of the positive definite matrix $\mathbf{M}$ (see Lay, 1994), divided by the upper-left element of that upper triangular matrix. At the t^{th} iteration, form

$$\hat{\mathbf{A}}_{dt} = \hat{\mathbf{A}}_{t-1} \hat{\mathbf{A}}_{t-2} \cdots \hat{\mathbf{A}}_0,$$

$$\mathbf{V}_{it} = \frac{\hat{\mathbf{A}}_{dt} \mathbf{Y}_i}{||\hat{\mathbf{A}}_{dt} \mathbf{Y}_i||} \quad \text{for} \quad i = 1, \cdots, n, \tag{A.1}$$

and

$$\mathbf{S}_t = n^{-1} \sum_{i=1}^{n} \mathbf{V}_{it} \mathbf{V}_{it}'.$$

If $||\mathbf{S}_t - p^{-1}\mathbf{I}||$ is sufficiently small, then stop and set $\hat{\mathbf{A}}_d = \hat{\mathbf{A}}_{dt}$. If not, then compute $\hat{\mathbf{A}}_t = \text{Chol}(\mathbf{S}_t^{-1})$ and go back to step (A.1).

APPENDIX B
UNDERLYING DISTRIBUTIONS AND RELATIONS AMONG THEM

As can be seen from the literature review that the assumption about the underlying distribution of the $\mathbf{Y}$'s varied from one test statistic to another. What follows is a brief definition of these distributional assumptions with an explanation about the relationships between them.

Parametric procedures usually have the most restrictive assumption, namely that the underlying population is a multivariate normal distribution. If $\mathbf{Y}_1, \cdots, \mathbf{Y}_n$ come from a multivariate normal distribution then each $\mathbf{Y}_i$ can be written as

$$\mathbf{Y}_i = R_i \mathbf{D} \mathbf{U}_i + \boldsymbol{\theta}, \tag{B.1}$$

where

- $\mathbf{U}_i$'s are iid uniformly distributed on the unit p-sphere,
- $R_i \overset{iid}{\sim} \sqrt{\chi^2(p)/p}$, independent of the $\mathbf{U}_i$'s, and
- $\mathbf{D}$ is a fixed nonsingular $p \times p$ matrix.

The family of multivariate normal distributions is a member of the larger class of elliptically symmetric distributions. In this less restrictive class, the representation in (B.1) still holds but now the R_i's are assumed to be iid positive random variables, not necessarily having a chi-square distribution but still independent of the $\mathbf{U}_i$'s. This class also includes, the power family, the family of multivariate t-distributions, the family of multivariate logistic distributions and many more.

Every member of the class of elliptically symmetric distributions is also a member of the class of distributions with elliptical directions. Distributions in the later class are also represented as in (B.1), but only require the R_i's to be positive numbers, not even iid or independent of the $\mathbf{U}_i$'s. Hence, this class includes skewed distributions since the R_i's can be large in some directions and small in others.

The class of elliptically symmetric distributions is also a subclass of the class of symmetric distributions, for which

$$(\mathbf{Y}_i - \boldsymbol{\theta}) \overset{d}{=} -(\mathbf{Y}_i - \boldsymbol{\theta}).$$

Thus, the probability at a point on any axis through $\boldsymbol{\theta}$ is the same as the probability at its reflection through $\boldsymbol{\theta}$ on that same axis.

A more general class is one whose distributions are such that the probability of being at one side of any axis through $\boldsymbol{\theta}$ is equal to the probability of being at the other side of that axis. Thus, if $\mathbf{Y}$ is a member of this class then

$$\frac{\mathbf{Y} - \boldsymbol{\theta}}{\|\mathbf{Y} - \boldsymbol{\theta}\|} \overset{d}{=} -\frac{\mathbf{Y} - \boldsymbol{\theta}}{\|\mathbf{Y} - \boldsymbol{\theta}\|}.$$

Distributions with this property are said to belong to the class of directionally symmetric distributions. It is clear that every symmetric distribution is directionally symmetric, and every distribution with elliptical directions is also directionally symmetric.

All the relationships between these various classes are presented in the figure B.1 (see next page).

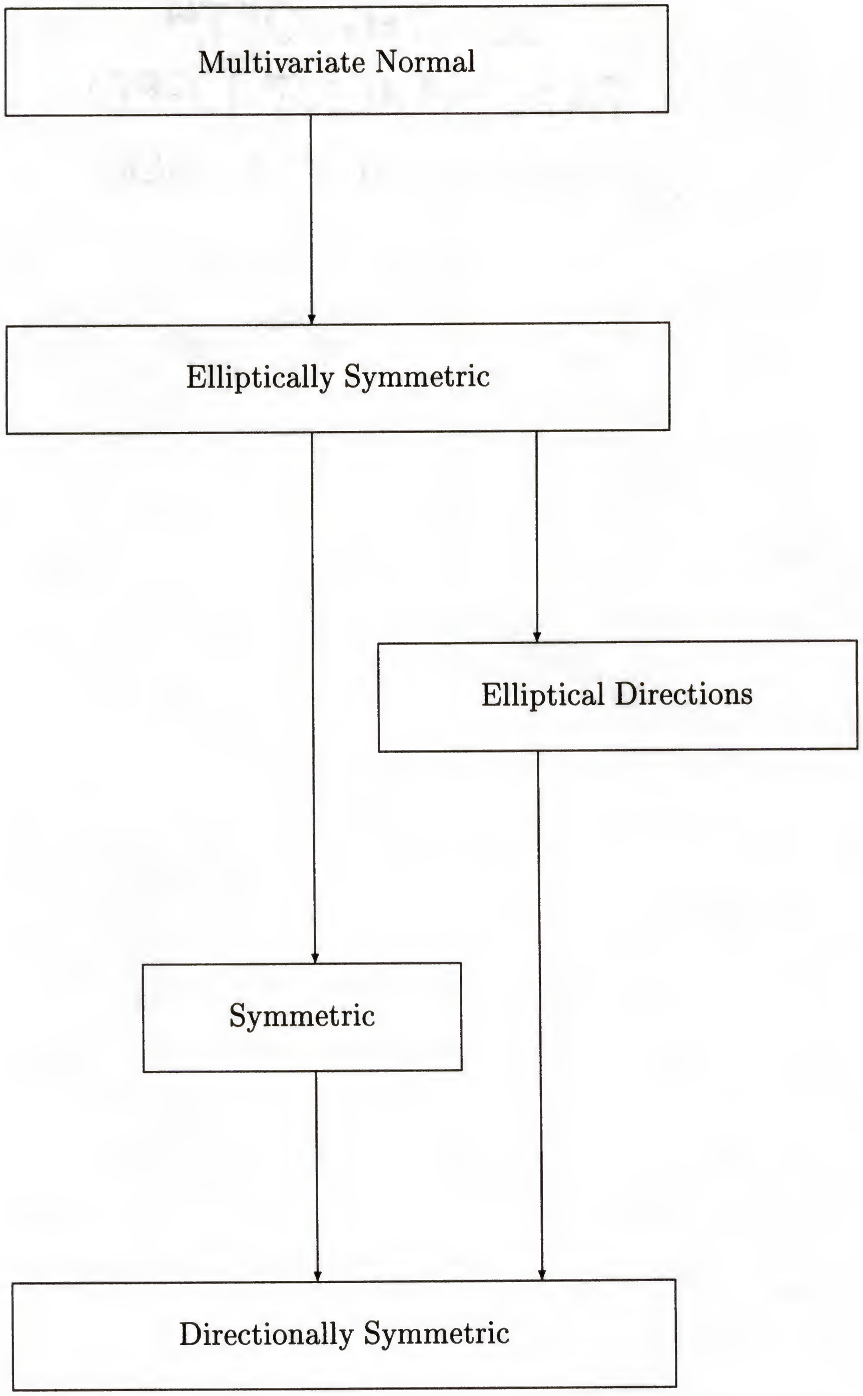

Figure B.1: Relationships Among Various Classes of Distributions

REFERENCES

Agresti, A. and Pendergast, J. (1986). Comparing mean ranks for repeated measures data, *Communications in Statistics, Part A – Theory and Methods* **15**: 1417–1433.

Bennett, B. M. (1962). On multivariate sign tests, *Journal of the Royal Statistical Society, Series B, Methodological* **24**: 159–191.

Bennett, B. M. (1964). A bivariate signed rank test, *Journal of the Royal Statistical Society, Series B, Methodological* **26**: 457–461.

Bickel, P. J. (1965). On some asymptotically non-parametric competitors of Hotelling's t^2, *The Annals of Mathematical Statistics* **36**: 160–173.

Billingsley, P. (1995). *Probability and Measure*, John Wiley & Sons, NewYork.

Blumen, I. (1958). A new bivariate sign test, *Journal of the American Statistical Association* **53**: 448–456.

Brown, B. M. and Hettmansperger, T. P. (1987). Affine invariant rank methods in the bivariate location model, *Journal of the Royal Statistical Society, Series B, Methodological* **49**: 301–310.

Brown, B. M. and Hettmansperger, T. P. (1989). An affine invariant bivariant version of the sign test, *Journal of the Royal Statistical Society, Series B, Methodological* **51**: 117–125.

Chakraborty, B., Chaudhuri, P. and Oja, H. (1998). Operating transformation retransformation on spatial median and angle test, *Statistica Sinica* **8**: 767–784.

Chatterjee, S. K. (1966). A bivariate sign test for location, *The Annals of Mathematical Statistics* **37**: 1771–1782.

Dietz, E. J. (1982). Bivariate nonparametric tests for the one-sample location problem, *Journal of the American Statistical Association* **77**: 163–169.

Dietz, E. J. (1984). Linear signed rank tests for multivariate location, *Communications in Statistics, Part A – Theory and Methods* **13**: 1435–1451.

Ernst, M. D. and Kepner, J. L. (1993). A Monte Carlo study of rank tests for repeated measures designs, *Communications in Statistics, Part B – Simulation and Computation* **22**: 671–678.

Fang, K.-T., Kotz, S. and Ng, K.-W. (1990). *Symmetric Multivariate and Related Distributions*, Chapman & Hall Ltd, London.

Friedman, M. (1937). The use of ranks to avoid the assumption of normality implicit in the analysis of variance, *Journal of the American Statistical Association* **32**: 675–701.

Gnedenko, B. V. (1962). *The Theory of Probability*, Chelsea Publishing Company, New York.

Greenhouse, S. W. and Geisser, S. (1959). On methods in the analysis of profile data, *Psychometrika* **24**: 95–112.

Hajek, J., Sidak, Z. and Sen, P. (1999). *Theory of Rank Tests*, Academic Press, San Diego.

Hannan, E. J. (1956). Non-linear time series regression (corr: 72v9 p686), *Journal of the Royal Statistical Society, Series B* **18**: 227–233.

Hettmansperger, T. P., Möttönen, J. and Oja, H. (1997). Affine-invariant multivariate one-sample signed-rank tests, *Journal of the American Statistical Association* **92**: 1591–1600.

Hettmansperger, T. P., Nyblom, J. and Oja, H. (1994). Affine invariant multivariate one-sample sign tests, *Journal of the Royal Statistical Society, Series B, Methodological* **56**: 221–234.

Hodges, J. L. (1955). A bivariate sign test, *Annals of Mathematical Statistics* **26**: 523–527.

Hora, S. C. and Iman, R. L. (1988). Asymptotic relative efficiencies of the rank-transformation procedure in randomized complete block designs, *Journal of the American Statistical Association* **83**: 462–470.

Hössjer, O. and Croux, C. (1995). Generalizing univariate signed rank statistics for testing and estimating a multivariate location parameter, *Journal of Nonparametric Statistics* **4**: 293–308.

Hotelling, H. (1931). The generalization of student's ratio, *Annals of Mathematical Statistics* **2**: 360–378.

Hsu, P. L. (1938). Notes on Hotelling's generalized t, *Annals of Mathematical Statistics* **9**: 231–243.

Huynh, H. and Feldt, L. S. (1970). Conditions under which mean square ratios in repeated measurements designs have exact f-distributions, *Journal of the American Statistical Association* **65**: 1582–1589.

Iman, R. L. and Davenport, J. M. (1980). Approximations of the critical region of the Friedman statistic, *Communications in Statistics, Part A – Theory and Methods* **9**: 571–595.

Iman, R. L., Hora, S. C. and Conover, W. J. (1984). Comparison of asymptotically distribution-free procedures for the analysis of complete blocks, *Journal of the American Statistical Association* **79**: 674–685.

Jan, S.-L. (1991). *Interdirection Tests for Repeated Measures and One-Sample Multivariate Location Problems*, PhD thesis, University of Florida, Fl.

Jan, S.-L. and Randles, R. H. (1994). A multivariate signed sum test for the one-sample location problem, *Journal of Nonparametric Statistics* **4**: 49–63.

Jan, S.-L. and Randles, R. H. (1996). Interdirection tests for simple repeated-measures designs, *Journal of the American Statistical Association* **91**: 1611–1618.

Jensen, D. R. (1977). On approximating the distributions of Friedman's x_r^2 and related statistics, *Metrika* **24**: 75–86.

Joffe, A. and Klotz, J. (1962). Null distribution and Bahadur efficiency of the Hodges bivariate sign test, *The Annals of Mathematical Statistics* **33**: 803–806.

Johnson, M. E. (1987). *Multivariate Statistical Simulation*, Wiley, New York.

Johnson, N. L. and Kotz, S. (1972). *Distributions in Statistics: Continuous Multivariate Distributions*, John Wiley & Sons, New York.

Kendall, M. and Babington, S. B. (1939). The problem of m ranking, *Annals of Mathematical Statistics* **10**: 275–287.

Kepner, J. L. and Robinson, D. H. (1988). Nonparametric methods for detecting treatment effects in repeated-measures designs, *Journal of the American Statistical Association* **83**: 456–461.

Killeen, T. J. and Hettmansperger, T. P. (1972). Bivariate tests for location and their Bahadur efficiencies, *The Annals of Mathematical Statistics* **43**: 1507–1516.

Klotz, J. (1964). Small sample power of the bivariate sign tests of Blumen and Hodges, *The Annals of Mathematical Statistics* **35**: 1576–1582.

Koch, G. G. (1969). Some aspects of the statistical analysis of "split plot" experiments in completely randomized layouts, *Journal of the American Statistical Association* **64**: 485–505.

Lay, D. C. (1994). *Linear Algebra and Its Applications*, Addison-Wesley Publishing Company, Massachusetts.

Morrison, D. F. (1972). The analysis of a single sample of repeated measurements, *Biometrics* **28**: 55–71.

Möttönen, J., Hettmansperger, T. P., Oja, H. and Tienari, J. (1998). On the efficiency of affine invariant multivariate rank tests, *Journal of Multivariate Analysis* **66**: 118–132.

Oja, H. (1983). Descriptive statistics for multivariate distributions, *Statistics & Probability Letters* **1**: 327–332.

Oja, H. and Nyblom, J. (1989). Bivariate sign tests, *Journal of the American Statistical Association* **84**: 249–259.

Peters, D. and Randles, R. H. (1990). A multivariate signed-rank test for the one-sampled location problem, *Journal of the American Statistical Association* **85**: 552–557.

Puri, M. L. and Sen, P. (1972). *Nonparametric Methods in Multivariate Analysis*, John Wiley & Sons, New York.

Quade, D. (1979). Using weighted rankings in the analysis of complete blocks with additive block effects, *Journal of the American Statistical Association* **74**: 680–683.

Randles, R. H. (1989). A distribution-free multivariate sign test based on interdirections, *Journal of the American Statistical Association* **84**: 1045–1050.

Randles, R. H. (2000). A simpler, affine-invariant, multivariate, distribution-free sign test, *Journal of the American Statistical Association* **95**: 1263–1268.

Randles, R. H. and Wolfe, D. A. (1991). *Introduction to the Theory of Nonparametric Statistics*, Krieger, Malabar.

Rudin, W. (1976). *Principles of Mathematical Analysis*, McGraw-Hill, New York.

Sen, P. K. and Puri, M. L. (1967). On the theory of rank order tests for location in the multivariate one sample problem, *The Annals of Mathematical Statistics* **38**: 1216–1228.

Serfling, R. J. (1980). *Approximation Theorems of Mathematical Statistics*, John Wiley & Sons, New York.

Silva, C. (1977). *Analysis of Randomixed Block Designs Based on Weighted Rankings*, PhD thesis, Chapel Hill, N.C.

Tyler, D. E. (1987). A distribution-free m-estimator of multivariate scatter, *The Annals of Statistics* **15**: 234–251.

Wilks, S. S. (1946). Sample criteria for testing equality of means, equality of variances, and equality of covariances in a normal multivariate distribution, *Annals of Mathematical Statistics* **17**: 257–281.

BIOGRAPHICAL SKETCH

Ziyad Mahfoud was born on August 4th, 1972 in Lebanon. He received a bachelor's degree in mathematics and a master's degree in mathematics from the American University of Beirut in 1993 and 1996, respectively. He came to the University of Florida in 1996 to pursue a doctorate degree in statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Ronald H. Randles

Ronald H. Randles, Chairman
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

P. V. Rao

Pejaver V. Rao
Professor Emeritus of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Malay Ghosh

Malay Ghosh
Distinguished Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James L. Kepner

James L. Kepner
Research Associate Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James Keesling

James Keesling
Professor of Mathematics

This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 2001

Dean, Graduate School